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Second Order Invexity and Duality In Non-Linear Programming Problem

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Abstract

In this paper, several kinds of second order invexity and duality problems are introduced in non-linear programming problem.

Keywords: Second order invexity, invex function, Type I function, generalized convexity and higher order duality.

1. Introduction

Consider the nonlinear programming problem:

$$\underset{x}{\text{Minimize}}\ f(x)$$

(P) subject to $g(x) \leq 0$,

where $x \in R^n$ and f and g are twice differentiable functions from R^n to R and R^m respectively.

The first order dual problem is:

(D1) $$\underset{u,\,y}{\text{Maximize}}\ f(u) + y'g(u)$$

$$\text{subject to } \nabla\{f(u) + y'g(u)\} = 0;$$

$$\text{and } y \geq 0,$$

where $y \in R^m$. By introducing an additional vector $p \in R^n$ Mangasanian [5] formulated the second order dual :

(D2) $$\underset{u,\,y,\,p}{\text{Maximize}}\ f(u) + y'g(u) - \frac{1}{2}p'\nabla^1[f(u) + y'g(u)]p$$

$$\text{subject to } \nabla[f(u) + y'g(u)] + \nabla^2[f(u) + y'g(u)]p = 0, \text{ and } y \geq 0,$$

Under appropriate conditions on f and g involving convexity, and complicated restrictions on p, duality theorems were established. Subsequently, Mond [6] gave simpler conditions than Mangasarian, using a generalized form of convexity. This type of generalization was also studied by Mahajan [3] and Mahajan and Varta [4], and a different form of second order duality was given by Mond and Weir [7]. Mond's generalization was extended to invexity by Bector. Chandra and Husain [1], by defining a class of functions which they called binvex. In this paper a further generalization is made and applied to duality.

The proofs of the duality theorems given here follow those of Mond [6]. One significant practical use of duality is that it provides bounds for the value of the objective function when approximations are used. Second order duality may provide tighter bounds than first order duality because there are more parameters involved.

This paper provides for applicability to a wider calss of functions and may give still tighter bounds than previously considered.

The dual is of the form:

$$\text{(D3)} \qquad \underset{u,\, y,\, p}{\text{Maximize}}\ f(u) + y' g(u) - \frac{1}{2} q' \nabla^2 [f(u) + y' g(u)]p$$

subject to $\nabla[f(u) + y'g(u)] + \nabla^2[f(u) + y'g(u)]p = 0$ and $y \geq 0$,

where $q \in R^n$ and $r \in R^n$. In general p, q and r can be regarded as functions, although the operators $\nabla$and ∇^2 in the above operate only on f and g. The constraints (1.3) – (1.4) are the same as in (D2).

2. Second Order Invexity:

Hanson and Mond [2] introduced a slight generalization of the class of invex functions to a class called Type I functions for use in mathematical programming. Here we generalize further to second order Type 1 functions.

Let K be the constraint set of (D3) given by (1.3) – (1.4) and let $\eta(x, u)$, $p(x, u)$, $q(x, u)$ and $r(x, u)$ be vector functions : $K \times K \rightarrow R^n$.

The objective function f(x) is said to be a second order Type 1 objective function and gi(x), i = 1, 2, ..., m is said to be a second order Type I constraint function at u € K with respect to the function $\eta(x,u)$, $p(x,u)$, $q(x,u)$ and $r(x,u)$ if for all x € K

$$f(x) - f(u) \geq \eta(x, u)' \nabla f(u) + \eta(x, u)' \nabla^2 f(u) p(x, u) - \frac{1}{2} q(x, u)' \nabla^2 f(u) r(x, u) \qquad \text{..... (2.1)}$$

and

$$-g_i(u) \geq \eta(x, u)' \nabla g_i(u) + \eta(x, u)' \nabla^2 g_i(u) p(x, u) - \frac{1}{2} q(x, u)' \nabla^2 g_i(u) r(x, u),\ i = 1, 2, ..., m \quad \text{.....(2.2)}$$

If p, q and r are all zero vectors, then (2.1) and (2.2) are the definitions of Type 1 functions given by Hanson and Mond [2].

3. Second Order Duality Theorems : Theorem 1. (Weak Duality)

Let x satisfy the constraints of (P) and u, y, p, r satisfy the constraints of (D3). Let f and gi, i = 1, 2, ..., m be second order Type I functions defined over the constraints sets of (P) and (D3).

Then infimum (P) $\geq$ supermum (D3)

Proof :

$$f(x) - f(u) - y'g(u) + \frac{1}{2}q(x,u)'\nabla^2[f(u) + y'g(u)]r(x,u) \geq \eta(x,u)'\nabla f(u) + \eta(x,u)'\nabla^2 f(u)p(x,u)$$

$$- \frac{1}{2}q(x,u)'\nabla^2 f(u)r(x,u) - y'g(u) + \frac{1}{2}q(x,u)\nabla^2[f(u) + yg(u)]r(x,u), \qquad \text{by (2.1)},$$

$$= \eta(x,u)'\nabla f(u) + \eta(x,u)'\nabla^2 f(u)p(x,u) - y'g(u) + \frac{1}{2}q(x,u)\nabla^2[y'g(u)]r(x,u),$$

$$= -\eta(x,u)'\nabla y'g(u) - \eta(x,u)'\nabla^2[y'g(u)] + p(x,u) - y'g(u)$$

$$+ \frac{1}{2}q(x,u)\nabla^2[y'g(u)]\, r(x,u), \quad \text{by(1.3)}$$

$$\geq 0, \text{ by (2.2) and (1.4)}$$

Theorem 2. (Duality)

Suppose x* is optimal in (P) and x* satisfies one of the usual constraint qualifications of mathematical programming, which makes the Kuhn-Tucker conditions at x* necessary conditions for a minimum. Then there exists $y \in R^m$ such that (x*y, p = q = r = 0) is feasible for (D3) and the corresponding values of (P) and (D3) are equal. If in addition (2.1) and (2.2) are satisfied for all feasible solutions of (D3) then x* and (x*, y, p = q = r = 0) are optimal for (P) and (D3) respectively.

Proof:

The Kuhn-Tucker conditions for a minimum at x* are that there exists $y \in R_m$ such that

$$\nabla f(x^*) + \nabla y'g(x^*) = 0$$

$$y'g(x^*) = 0 \text{ and } y \geq 0$$

so that point (x*, y, p = q = r = 0) is feasible for (D3) and the values of (P) and (D3) are equal, and it follows from Theorem 1 that x* and (x*, y, p = q = r = 0) are optimal for (P) and (D3).

Since p = q = r = 0 at optimum, in which case the second order dual reduces to the first order dual, there may seem to be no point in having the additional complication of

introducing the extra functions p, q and r in the approximating dual are not necessarily zero, and the second order dual may be used to give a tighter bound than the first order dual for the value of the primal objective function.
Note that in the proof of Theorem 2, it would be sufficient for either q or r to be zero and not necessarily both.

Consider the problem :

Minimize $f(x_1, x_2) \equiv x_1x_2+x_1+x_2$ ... (3.1)

subject to $g_1(x_1, x_2) \equiv x_1^3 - x_2 \le 0,$ ... (3.2)

$g_2(x_1, x_2) \equiv -x_1x_2 - 1 \le 0,$ ... (3.3)

and $g_3(x_1, x_2) \equiv -x_1 + x_2^2 \le 0,$ ... (3.4)

we have

$$\nabla f = \begin{bmatrix} x_2+1 \\ x_1+1 \end{bmatrix}, \nabla^2 f = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\nabla g_1 = \begin{bmatrix} 3x_1^2 \\ -1 \end{bmatrix}, \nabla^2 g_1 = \begin{bmatrix} 6x_1 & 0 \\ 0 & 0 \end{bmatrix}$$

$$\nabla g_2 = \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}, \nabla^2 g_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$\nabla g_3 = \begin{bmatrix} -1 \\ 2x_2 \end{bmatrix}, \nabla^2 g_3 = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

Note that (3.4) implies that $x_1 \ge 0$ and hence (3.2) implies that $x_1 \ge 0$. The minimal value is clearly 0 at the point $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$.
Suppose we do not know this, and desire to examine the value of an approximate solution.
To illustrate second order duality let us compare a lower bound to the minimal value given by this approximation ($u_1 = 1$, $u_2 = 1$) in problem (D3) with a lower bound in (D1). The value of (P) at point $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is 3,
which is thus an upper bound to the true optimal value. However, as follows, note that (D1) does not have a feasible solution; so it does not provide a lower bound.

At $\begin{bmatrix}1\\1\end{bmatrix}$ the constraints of (D1) are $\nabla[f(u)+y'g(u)]=0$ and $y \geq 0$

i.e., $2+3y_1+y_2-y_3=0,$ ... (3.5)

$2-y_1+y_2+2y_3=0,$ ... (3.6)

and $y_1, y_2, y_3 \geq 0$

Subtracting (3.5) from (3.6) we have $-4y_1+3y_3=0 \Rightarrow y_3=\frac{4}{3}y_1$ ... (3.7)

and substituting this value in (3.6) we have

$$2+y_2+\frac{5}{3}y_1=0,$$

which contradicts (3.7) and thus there is no feasible solution for (D1).

We now find expressions for y, η, p, q and r for which the functions f, g_1, g_2, g_3 are second order Type 1 and satisfy the constraints of (D3). Such expressions are not necessarily unique and in this paper are assigned rather arbitrary although it happens that we find a set that is best possible. In general, finding such a set could be a mathematical programming problem in itself, but since we are looking for bounds such a formality can be avoided, and from a practical point of view, such a set need not be best possible.

4. Conditions for f, g_1, g_2, g_3 to be second order Type I at $u_1=1$, $u_2=1$.

(a) For f(x);

$$f(x)-f(u) \geq \eta(x,u)'\nabla f(u)+\eta(x,u)'\nabla^2 f(u)p(x,u)-\frac{1}{2}q(x,u)'\nabla^2 f(u)r(x,u),$$

i.e., $x_1x_2+x_1+x_2-3 \geq [\eta_1 \quad \eta_2]\begin{bmatrix}2\\2\end{bmatrix}+[\eta_1 \quad \eta_2]\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix}-\frac{1}{2}[q_1 \quad q_2]\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}r_1\\r_2\end{bmatrix}$

where the arguments (x, u) have been omitted for notational convenience. Since $x_1 \geq 0$ and $x_2 \geq 0$, we require at most

(a) For f(x);

$$-3 \geq 2\eta_1+2\eta_2+\eta_2 p_1+\eta_1 p_2-\frac{1}{2}(q_2 r_1+q_1 r_2) \quad \text{... (4.1)}$$

(b) For $g_1(x)$;

$$-g_1(u) \geq \eta(x,u)'\nabla g_1(u) + \eta(x,u)'\nabla^2 g_1(u)p(x,u) - \frac{1}{2}q(x,u)'\nabla^2 g_1(u)r(x,u) \qquad \text{... (4.2)}$$

i.e., $0 \geq [\eta_1 \quad \eta_2]\begin{bmatrix}3\\-1\end{bmatrix} + [\eta_1 \quad \eta_2]\begin{bmatrix}6 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix} - \frac{1}{2}[q_1 \quad q_2]\begin{bmatrix}6 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}r_1\\r_2\end{bmatrix}$

So, $0 \geq 3\eta_1 - \eta_2 + 6\eta_1 p_1 - 3q_1 r_2$ ... (4.3)

(c) For $g_2(x)$;

$$-g_2(u) \geq \eta(x,u)'\nabla g_2(u) + \eta(x,u)'\nabla^2 g_2(u)p(x,u) - \frac{1}{2}q(x,u)'\nabla^2 g_2(u)r(x,u),$$

i.e., $0 \geq [\eta_1 \quad \eta_2]\begin{bmatrix}1\\1\end{bmatrix} + [\eta_1 \quad \eta_2]\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix} - \frac{1}{2}[q_1 \quad q_2]\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}r_1\\r_2\end{bmatrix}$

So, $0 \geq \eta_1 + \eta_2 + \eta_2 p_1 + \eta_1 p_2 - \frac{1}{2}(q_2 r_1 + q_1 r_2)$ ... (4.4)

(d) For $g_3(x)$;

$$-g_3(u) \geq \eta(x,u)'\nabla g_3(u) + \eta(x,u)'\nabla^2 g_3(u)p(x,u) - \frac{1}{2}q(x,u)'\nabla^2 g_3(u)r(x,u)$$

i.e., $0 \geq [\eta_1 \quad \eta_2]\begin{bmatrix}-1\\2\end{bmatrix} + [\eta_1 \quad \eta_2]\begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix} - \frac{1}{2}[q_1 \quad q_2]\begin{bmatrix}0 & 0\\0 & 2\end{bmatrix}\begin{bmatrix}r_1\\r_2\end{bmatrix}$

So, $0 \geq -\eta_1 + 2\eta_2 + 2p_2 - q_2 r_2$ ... (4.5)

5. Conditions for f, g_1, g_2, g_3 to satisfy the constraints of (D3) at $u_1 = 1$, $u_2 = 1$.

$\nabla[f(u)+y'g(u)] + \nabla^2[f(u)+y'g(u)]\,p(x,u) = 0$

and $u \geq 0$

i.e., $\begin{bmatrix}2\\2\end{bmatrix} + y_1\begin{bmatrix}3\\-1\end{bmatrix} + y_2\begin{bmatrix}1\\1\end{bmatrix} + y_3\begin{bmatrix}-1\\2\end{bmatrix} + \begin{bmatrix}6y_1 & 1+y_2\\1+y_2 & 2y_3\end{bmatrix}\begin{bmatrix}p_1\\p_2\end{bmatrix} = 0$

and $y_1, y_2, y_3 \geq 0$

$\Rightarrow 2 + 3y_1 + y_2 - y_3 + 6y_1p_1 + (1+y_2)p_2 = 0,$ ... (4.6)

$2 - y_1 + y_2 + 2y_3 + (1+y_2)p_1 + 2y_3p_2 = 0,$ ... (4.7)

and $y_1, y_2, y_3 \geq 0$ ... (4.8)

If we put $p_1 = -\frac{1}{2}$ and $p_2 = 1$, we have from (4.2) and (4.3)

$3 + 2y_2 - y_3 = 0,$

$\frac{3}{2} - y_1 + \frac{1}{2}y_2 + 4y_3 = 0,$

and $y_1, y_2, y_3 \geq 0$

for which one solution is $y_1 = \frac{27}{2}, y_2 = 0, y_3 = 3.$

At $u_1 = 1, u_2 = 1$ the objective function in (D3) has the value

$$f(u) + y' g(u) - \frac{1}{2}[q_1 \quad q_2]\nabla^2 [f(u) + y' g(u)]\begin{bmatrix} r_1 \\ r_2 \end{bmatrix},$$

$$= 3 + y_1(0) + y_2(0) + y_3(0)$$

$$- \frac{1}{2}[6q_1y_1r_1 + q_2(1+y_2)r_1 + q_1(1+y_2)r_2 + 2q_2y_3r_2],$$

$$= 3 - [81q_1r_1+q_2r_1+q_1r_2+6q_2r_2] = 0$$

If we put $q_1 = 0, q_2 = 1, r_1 = 0, r_2 = 1,$

it remains to be shown that with these values of p, q and r the functions f, g_1, g_2 and g_3 are second order Type 1 functions.

The conditions (4.2), (4.3), (4.4) and (4.5) become

(a) $-3 \geq 3\eta_1 + \frac{3}{2}\eta_2$

(b) $0 \geq \eta_2$

(c) $0 \geq 2\eta_1 + \frac{1}{2}\eta_2$

(d) $0 \geq -\eta_1 + 4\eta_2 - 1$

which are all satisfied if we put $\eta_1 = -1$ and $\eta_2 = 0$.

So f, g, g and g are second order Type I function at $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$.

Since the optimal value of (P) is 0 and the objective value of (D3) at $u_1 = 1$, $u_2 = 1$ is also 0 for the values of η, y, p, q and r that we have obtained, these values are best possible by Theorem 1, though not necessarily unique. Of course in a general problem we would not know the optimal value of the primal problem, and would not know if the value of η, y, p, q, and r best possible, but any set of values satisfying the conditions imposed will give a lower bound for the optimal value of the primal problem.

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