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Topologies on Dual Spaces and Spaces of Linear Mappings

Dr.P.A.S.Naidu
D.B.Science College, Gondia(Maharashtra)

B.G.Mapari
PhD Scholar,R.T.M.Nagpur University

P. Jha
Govt.J.Y.Chhattisgarh College, Raipur (C.G.)

Abstract:

Let X and Y be two ***convex spaces*** over the same (real or complex) field F. We consider a general method of defining ***convex topologies*** on the dual of a convex space, taking as neighborhoods of the origin the ***polars*** of certain sets in the convex space. Here we have proved that any finite sum of compact sets is compact. Also the sum of a compact and a closed set in a convex space is closed. Let V be the vector space of all continuous linear mappings of X into Y. Let A be any bounded subsets of X and B a base of ***absolutely convex neighborhoods*** in Y. Define $W_{A,\,B}$= {t: t (A) $\subseteq$ B} for each A$\in$A and B$\in$B. Then $W_{A,\,B}$ is absolutely convex and ***absorbent***. The topology is then called the ***topology of A –convergence***. If X is a ***barreled space***, then any point wise bounded set of continuous linear mappings of X into Y is ***equicontinuous***.

1. Introduction and Necessary Preliminaries

A ***topological vector space*** (tvs) is a set with two compatible structures, one, it has the algebraic structure of the vector space, and the other, it has a topology so that the notions of convergence and continuity are meaningful.

A subset V of a tvs X is ***convex*** if λ x + μ y $\in$V for all x , y $\in$ V whenever $\lambda + \mu = 1$. V is ***balanced***if x$\in$V for all x $\in$ V whenever $|\lambda| \leq 1$. V is called ***absolutely convex*** if it is both convex and balanced. The set of all finite linear combinations $\sum_i \lambda_i x_i$ with

$\lambda_i \geq 0, \sum_i \lambda_i = 1$ and each $x_i \in V$ is called ***convex envelope*** of A. The ***absolutely convex envelope*** of V is the set of all finite linear combinations $\sum_i \lambda_i x_i$ with $\sum_i |\lambda_i| \leq 1$ and each $x_i \in V$ and is the smallest convex set containing V. The set V is ***absorbent*** if x ∈ V there is some λ> 0 such that x ∈μV for all μ with $|\mu| \geq \lambda$.

A topology ξ on X is said to be ***compatible*** with the algebraic structure of X if the algebraic operations '+' and'.' are continuous in X.

A topological vector space is ***locally convex*** if it has a base of convex neighborhood of the origin.

In a convex space, a subset is called a ***barrel*** if it is absolutely convex, absorbent and closed. A convex space is called barreled if every barrel is a neighborhood.
A non-negative real valued function p on a tvs X is called a ***semi norm***
if (i) p(x) ≥ 0;
(ii) $p(\lambda x) = |\lambda|$ p(x);
(iii) $p(x+y) \leq p(x) + p(y)$

In addition if p(x)=0 implies x = 0, then p is called a ***norm***.
The semi norm p corresponding to the absolutely convex absorbent set V defined by p(x) = inf $\{\lambda : \lambda > 0, x \in \lambda V\}$ along with the property that

$$\{x : p(x) < 1\} \subseteq V \subseteq \{x : p(x) \leq 1\}$$

is called the ***gauge*** of V.

The space X is called ***normable*** if its topology can be determined from a norm p.
The ***dual*** of a vector space is the vector space of all continuous linear mappings of the space into the scalar field.

A topological space is ***compact*** if its each open cover has a finite sub cover.
A topological space X is said to be ***supercompact*** if there is a subbase $\wp$ for its open sets such that each open cover of X by elements of $\wp$ have a sub cover consisting of at most two elements of $\wp$.

2. General Method of Defining Convex Topologies and Compactness

Let (X, X′) be a dual pair and ***A*** any set of weakly bounded subsets of X. Then the sets A^0 (A ∈***A***), where is A^0 called a ***polar*** of A given by

$$A^0 = \{x' : \sup\{|\langle x, x'\rangle| : x \in A\} \leq 1\}$$

is absolutely convex and absorbent and so there is a coarsest topology τ' on X′ in which they are neighborhoods. A base in neighborhoods in τ' is formed by the sets

$$\varepsilon \bigcap_{1\le i\le n} A_i^0 = \left(\varepsilon^{-1} \bigcup_{1\le i\le n} A_i\right)^0 \quad (\varepsilon > 0, A_i \in \mathcal{A})$$

This topology τ' on X′ is called the topology of uniform convergence on the sets of ***A*** or the topology of ***A***- convergence or the ***polar topology***.

Now suppose X is a vector space over the field of real or complex numbers. Let U be a non-empty set of subsets of X such that

(i) if U ∈ U , V∈ U , there is a W ∈ U with W⊆U∩V;

(ii) if U ∈ U and $\alpha \neq 0, \alpha U \in$ U;

(iii) each U ∈ U is absolutely convex and absorbent.

Then there exists a topology τ making X a convex space with U as a base of neighborhoods. This topology τ on X is called the ***convex topology***.

Theorem 2.1 Any finite sum of compact sets in a convex space is compact.

Proof: Let A and B be two compact sets and C an open covering of A + B. Then for each x ∈ A and each y ∈ B, there is an open absolutely convex neighborhood U(x,y) of the origin for which x + y + U(x,y) is contained in some set of C . Now keeping x fixed, the sets y + ½ U(x,y) form an open covering of B. As B is compact, let $\left\{y_j + \frac{1}{2}U(x, y_j) : 1 \le j \le n(x)\right\}$ be finite sub cover of y + ½ U(x,y). Let $V(x) = \bigcap_{1\le j\le n(x)} \frac{1}{2}U(x, y_j)$ Then the sets x +V(x) form an open covering of A. Again as A is compact, let $\{x_i + V(x_i) : 1 \le i \le m\}$ be a finite sub covering of A. Then

$$A + B \subseteq \bigcup_{1\le i\le m} (x_i + V(x_i))$$

$$\subseteq \bigcup_{1\le i\le m} \bigcup_{1\le j\le n(x_i)} \left(x_i + y_j + \frac{1}{2}U(x_i, y_j) + \frac{1}{2}U(x_i, y_j)\right)$$

$$\subseteq \bigcup_{1\le i\le m} \bigcup_{1\le j\le n(x_i)} (x_i + y_j + U(x_i, y_j))$$

Hence A + B is compact.

Theorem2.2 The sum of a compact set and a closed set in a convex space is closed.

Proof: Let A be compact and B closed. Let $a \notin A + B$. Then for each x ∈ A, x + B is closed, for + is continuous and B is closed. Hence there is an absolutely convex neighborhood U(x) of the origin with (a + U (x)) ∩(x + B) = ϕ. Then $a \notin x + U(x) + B$. Now $\left\{x + \frac{1}{2}U(x)\right\}_{x\in A}$ form an open cover of A. Since A is compact, let $\left\{x_i + \frac{1}{2}U(x_i)\right\}_{x_i \in A}$, $1 \le i \le n$, form a finite sub cover of A. Let $V = \bigcap_{1 \le i \le n} \frac{1}{2}U(x_i)$.

Then

$$A + V \subseteq \bigcup_{1 \le i \le n}\left(x_i + \frac{1}{2}U(x_i) + \frac{1}{2}U(x_i)\right)$$

$$\subseteq \bigcup_{x \in A}(x + U(x))$$

Hence $a \notin A + V + B$. Thus

(A+V)∩(A+B)= ϕ

and so $a \notin \overline{(A+B)}$. Hence A + B is closed.

If X is separated convex space, then the set of compact subsets of X can be used to define a polar topology on X′. Now if A is bounded, then the absolutely convex envelope of A is also bounded. However, it is not true in general that the closed absolutely convex envelope of a compact set is compact.

Remark: One may check whether the closed absolutely convex envelope of a super compact set is super compact or not?

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