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Three-Space Stability of Various Reflexivities in Locally Convex Spaces

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Abstract

In this paper, we consider three-space-problem in locally convex spaces for various types of “property (P)”, like- polar semi-reflexivity, polar reflexivity, semi-reflexivity, reflexivity, inductive semi-reflexivity, inductive reflexivity, B-semireflexivity, B-reflexivity. Our purpose is to harmonize several known positive solutions to three-space-problem and to investigate for some new results.

Keywords: Three-space-problem, polar reflexive, reflexive, Inductively reflexive, B-reflexive, bornological.

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1. Introduction and Preliminaries

Let $E[\tau]$ be a locally convex space and F a (closed) subspace such that F and the corresponding quotient E/F possess a certain property P; does E also possess P?. This so called problem is called a **three-space-problem**. If the answer to such a question is positive (affirmative), then we say that the property P is three-space-stable property or simply three-space property.

This problem can be put in terms of a short-exact sequence as follows:
Le $E[\tau]$ be a locally convex space and F is a (closed) subspace of E.
If a short exact sequence

$$o \rightarrow F \xrightarrow{i} E \xrightarrow{q} E/F \rightarrow o \qquad (1)$$

of locally convex space E, a subspace F of E and the corresponding quotient E/F, is given in which the outer terms F and E/F possess certain property (P), whether the middle term, space E possess the property (P)?.

Throughout this paper, $E[\tau]$ will represent a locally convex Hausdorff topological vector space (abbreviated as locally convex space) over K (real or complex field). Its dual is denoted as E'. The strong dual of $E[\tau]$ is $E'[\tau_b(E)]$ and the bidual of $E[\tau]$ is $E'' = (E'[\tau_b(E)])'$. If $E''= E$, then $E[\tau]$ is called semi-reflexive. A semi-reflexive locally convex space $E[\tau]$ is called reflexive provided $\tau = \tau_b(E')$. Following G. Köthe [1], let τ^o be the topology on E' of uniform convergence on τ-precompact subsets of E and τ^{oo}be the topology on $(E'[\tau^o])'$ of uniform convergence on τ^o-precompact subsets of E'. If $(E'[\tau^o])' = E$, then $E[\tau]$ is called polar semi-reflexive, and polar reflexive if further $\tau = \tau^{oo}$. The so called p-complete spaces and p-reflexive spaces of Brauner [2] are nothing but polar semi-reflexive and polar reflexive, respectively (see [3]). We also note that polar reflexivity is the t-reflexivity of Kye [4].

The strongest locally convex topology on E' for which all τ-equicontinuous subsets are bounded is denoted by τ^*, called inductive topology. The absolutely convex subsets of E' that absorbs all τ-equicontinuous subsets of E' form a basis of neighborhoods of 0 in $E'[\tau^*]$. According to [5], if $(E'[\tau^*])'$ coincides with E, then $E[\tau]$ is called inductively semi-reflexive. Moreover, if $\tau = \tau^{**}$ i.e. $(\tau^*)^*$, then $E[\tau]$ is called inductively reflexive.

Let $\boldsymbol{\tau_r}$ be the topology, called reflective topology, on E of uniform convergence over the smallest saturated class of sets generated by $\mathcal{R}$, where $\mathcal{R}$ is the class of all the absolutely convex bounded subset B of the dual E' whose span space E'_B is a reflexive Banach space with B as unit ball; such sets B are called reflective sets. A locally convex space $E[\tau]$ is said to be B-semireflexive if it is barreled and $E = \tilde{E}[\tilde{\tau}_r]$ (completion of $E[\tau_r]$). Further, if $\tau = \tau_r$, then $E[\tau]$ is called B-reflexive [6].

2. Three-Space Stability of Various Reflexivities

Let E be a Banach space, E' and E'' its first and second dual spaces. let Q denote the canonical embedding of E in to E'' . Banach space E is called reflexive if the canonical embedding Q is an onto mapping. If $\dim (E''/Q(E)) = n$ (finite), then we say that E is quasi-reflexive of order n. If $E''/Q(E)$ is reflexive, then E is called coreflexive.The so called quotient reflexive spaces of Yorke [7] are nothing but coreflexive spaces.

In [8], James R. Clark, proved that coreflexivity in Banach spaces is three-space stable:

2.1. Theorem ([8]): If F is a closed subspace of a Banach space E , then E is coreflexive if and only if F and E/F are coreflexive.

P. Civin and B. Yood [9] discussed three-space-problem on quasi-reflexivity in Banach spaces and proved the following:

2.2. Theorem ([9]): If F is a closed subspace of a Banach space E, then E is quasi-reflexive if and only if F and E/F are quasi-reflexive.

We know that if $E[\tau]$ is a reflexive Banach space, then its closed subspace F and separated quotient E/F are both reflexive [1]. In the next theorem of Kung-Wei Yang, in [10], in which he discussed three-space stability of reflexivity in Banach spaces, we see that the converse of this result is also true.

2.3 Theorem ([10]): If E is a Banach space and F is a closed subspace of E, and if both F and E/F are reflexive then E is reflexive.

From these theorems it is obtained that in Banach spaces, each of coreflexivity, quasi-reflexivity and reflexivity is a three-space stable property.

Let $E[\tau]$ be a locally convex space. If τ_c (resp. τ_{b^*}) is the topology on E' of uniform convergence on the class of compact disks (resp. strongly bounded subsets) of $E[\tau]$, and if τ_{cc} (resp. $\tau_{b^{**}}$) is the topology on $(E'[\tau_c])'$ (resp. $(E'[\tau_{b^*}])'$) of uniform convergence on the class of τ_c -compact disks in $E'[\tau_c]$ (resp. strongly bounded subset in $E'[\tau_{b^*}]$), then $E[\tau]$ is called c-semi-reflexive if $E = (E'[\tau_c])'$ and further, if $\tau = \tau_{cc}$,$E[\tau]$ is called c-reflexive. It is clear that $E=(E'[\tau_c])'$ always holds, since (by definition) τ_c is coarser than the Mackey topology. So $E[\tau]$ is always c-semi-reflexive. A locally convex space $E[\tau]$ is called b*-semireflexive if $(E'[\tau_{b^*}]))' = E$. Further, if $\tau = \tau_{b^{**}}$, it is called b*-reflexive.

Following theorem on three-space stability of c-reflexive spaces is due to [11]:

2.4. Theorem B ([11]): Let F be a closed subspace of a locally convex space $E[\tau]$ such that F and E/F are c-reflexive, then $E[\tau]$ is c-reflexive, provided the quotient map from E to E/F lifts compact disks.

We also have: In a locally convex space, b*-semireflexivity and b*-reflxivity are not three-space stable ([12], Theorem-11).

It is to be mentioned here, that polar semi-reflexivity is not three-space stable ([12],Theorem-9).
In locally convex spaces, completeness is a three-space stable property. That is, if $E[\tau]$ is a locally convex space and F is a closed subspace of E such that F and E/F are complete, then $E[\tau]$ is complete (see [13]).

W. Roelcke and S. Dierolf[14] established some results on three-space-problem. They investigated that barreledness and ultra barreledness are three-space-stable properties. Further, if F is a subspace of a locally convex space $E[\tau]$ such that F is quasi-barreled and E/F is barreled, then E is quasi-barreled (Theorem 2.6). Regarding semi-reflexivity property, we have:

2.5. Theorem ([14]): Let $E[\tau]$ be a locally convex space and F a closed linear subspace such that the strong topology $\tau_b(F)$ on $E'/F^{\perp}$ = the quotient topology on $E'/F^{\perp}$ induced from $\tau_b(E)$. That is $\tau_b(F, E'/F^{\perp}) = (\tau_b(E))_q F^{\perp}$. If F and the quotient E/F are semi-reflexive, then $E[\tau]$ is semi-reflexive.
Proof: Assume that $\tau_b(F , E'/F^{\perp}) = (\tau_b(E))_q F^{\perp}$ and the subspace F and the quotient E/F both are semi-reflexive. Let $f \in (E'[\tau_b(E)])'$. Since $\tau_b(E/F, F^{\perp})$ on $F^{\perp}$ is finer than the relative topology from $\tau_b(E)$, so the restriction of f on $F^{\perp}$ is continuous on $F^{\perp}[\tau_b(E/F, F^{\perp})]$. We note that $(E/F)' = F^{\perp}$ and is given semi-reflexive, so we have $x \in E$ such that $f(u) = u(x)$ for all $u \in F^{\perp}$. Now $G: (E'[\tau_b(E)]) \rightarrow K$, $G(u) = f(u)$ is a continuous linear functional which vanishes on $F^{\perp}$. Thus G induces a continuous linear functional $G^{\sim}$ on $E'/F^{\perp} [(\tau_b(E))_q F^{\perp}]$. From hypothesis, $\tau_b(F , E'/F^{\perp}) = (\tau_b (E))_q F^{\perp}$, so by identifying $F' = E'/F^{\perp}$ and using the semi-reflexivity (given) of F, we must have $y \in F$ such that $G(u) = G^{\sim}(u+ F^{\perp}) = u(y)$ for all $u \in E'$. It follows that $f(u) = u(x + y)$ for all $u \in E'$. Hence $f \in E$. Thus $E'' = E$ and $E[\tau]$ is semi-reflexive.

For a general locally convex space, semi-reflexivity is not three-space stable (see [14], Example-1.5)

Now we have the following result on three-space stability of semi-reflexivity:

2.6. Theorem ([14]): If $E[\tau]$ is a locally convex space and F is a closed subspace of E which is an (F)-space. If F and E/F are semi-reflexive, then $E[\tau]$ is semi-reflexive.

Proof: Assume that F and E/F are semi-reflexive. Since F is an (F)-space, so it is countably quasi-barreled and its strong dual is bornological. Therefore, by [14], lemma-4.1, $\tau_b(F , E'/F^{\perp}) = (\tau_b(E))_q F^{\perp}$ holds. By theorem-2.5, the proof is complete.

We know that inductively semi-reflexive space is semi-reflexive. Further, if E[τ] is an (F)-space, then it is inductively semi-reflexive if and only if it is semi-reflexive (see [5]).
Now we assert the followong:

2.7. Theorem: If F is a closed subspace of an (F)-space E[τ] such that F is also (F)-space, and if F and E/F are inductively semi reflexive, then E[τ] is inductively semi reflexive.

Proof: Assume that F and E/F are inductively semi reflexive. Since inductively semi-reflexive space is semi-reflexive, F and E/F are semi reflexive. Further, subspace F is an (F)-space, so theorem-2.6 is applicable and so E[τ] is semi-reflexive. Since semi-reflexive (F)-space is reflexive and a reflexive (F)-space is inductively reflexive, E[τ] is inductively reflexive (and inductively semi-reflexive).

2.8. Corollary : If F is a closed subspace of an (F)-space E[τ] such that F is also an (F)-spaces, and if F and E/F are inductively reflexive, then E[τ] is inductively reflexive.

These results on three-space stability of inductive (semi) reflexivity are restricted to the case where the subspace F is an (F)-space. However we have a generalization in the following:

2.9. Theorem ([16]): If F is a closed subspace of a locally convex space E[τ], and if E and E/F are inductively semi-reflexive, then E[τ] is inductively semi-reflexive.
Proof: Consider the short exact sequence

$$o \rightarrow F \xrightarrow{\ i\ } E \xrightarrow{\ q\ } E/F \rightarrow o \qquad (1)$$

Let us consider the inductive topology τ^* on E′. Note that $F' = E'/F^{\perp}$ and $(E/F)' = F^{\perp}$.

Let $\hat{\tau}$= relative topology on F induced from E[τ], τ_q = quotient topology on E/F, $\hat{\tau}^*$ = the inductive topology on F′ (the dual of F[$\hat{\tau}$]), and τ_q^* = the inductive topology on (E/F)′ (the dual of E/F[τ_q]). Now we prove some lemmas.

Lemma-1: If F is a closed subspace of a locally convex space E[τ], then $\hat{\tau}^*$ is equal to the quotient topology $(\tau^*)_{q(E'/F^{\perp})}$ on $E'/F^{\perp}$ from E′[τ^*]. i.e. $\hat{\tau}^* = (\tau^*)_{q(E'/F^{\perp})}$.

Proof: We prove $\hat{\tau}^* \geq (\tau^*)_{q(E'/F^{\perp})}$. Since $\hat{\tau}^*$ is the strongest locally convex topology on F′ such that all $\hat{\tau}$–equicontinuous set in F′ are bounded. Therefore, it is sufficient to prove that all $\hat{\tau}$– equicontinuous set in F′ are $(\tau^*)_{q(E'/F^{\perp})}$-bounded. let

$A \subset F'$ be a $\tau^{\wedge}$-equicontinuous subset. That is $A = i'(B)$, where $B \subset E'$ is τ-equicontinuous. Since B is τ^*-bounded , A is $(\tau^*)_q(E'/F^{\perp})$-bounded. So $\tau^{\wedge *} \geq (\tau^*)_q(E'/F^{\perp})$.

We prove $\tau^{\wedge *} \leq (\tau^*)_q(E'/F^{\perp})$. Let W be a $\tau^{\wedge *}$ -neighborhood of 0. Then W absorbs all $\tau^{\wedge}$–equicontinuous subset of F' and so $(i')^{-1}(W)$ is a τ^*- neighborhood of 0. So W is a $(\tau^*)_q(E'/F^{\perp})$ -neighborhood of 0. Hence $\tau^{\wedge *} \leq (\tau^*)_q(E'/F^{\perp})$.

Lemma-2: If F is a closed subspace of a locally convex space $E[\tau]$, then the relative topology $(\tau^*)^{\wedge}$ on $F^{\perp}$ induced from $E'[\tau^*]$ is coarser than τ_q^* i.e. $(\tau^*)^{\wedge} \leq \tau_q^*$.

Proof: Let V be a $(\tau^*)^{\wedge}$ -neighborhood of 0. By definition of induced topology, we have a τ^* –neighborhood of 0, say U such that $U \cap F^{\perp} \subset V$. So $U \cap F^{\perp}$ absorbs all $\tau^{\wedge}$ -equicontinuous subset of $F^{\perp}$ and so it is a τ_q^*-neighborhood of 0. Hence $(\tau^*)^{\wedge} \leq \tau_q^*$.

Lemma-3 : Following relations among topologies on the space $E'/F^{\perp}$ hold:
$\tau_k(F , E'/F^{\perp}) = \tau_b(F , E'/F^{\perp}) = \tau^{\wedge *} = (\tau^*)_{q(E'/F^{\perp})} \geq (\tau_b(E))_q \geq \tau_b(F)$.

Proof: Since F is inductively semi-reflexive, $(F'(\tau^{\wedge *}))' = F$. So $\tau^{\wedge *}$ is compatible to the dual pair (F', F). Hence $\tau_k(F , E'/F^{\perp}) = \tau_b(F , E'/F^{\perp}) = \tau^{\wedge *}$. On the other hand, $\tau^{\wedge *} = (\tau^*)_{q(E'/F^{\perp})}$ by lemma-1, and $(\tau^*)_{q(E'/F^{\perp})} \geq (\tau_b(E))_q$ by the fact that $\tau_b(E)$ is always coarser than τ^*, and $(\tau_b(E))_q \geq \tau_b(F)$ by [1], §22,2(4)-line 8 in its proof.

Lemma-4 : In the subspace $F^{\perp}$ we have:
$\tau_k(E/F, F^{\perp}) = \tau_b(E/F, F^{\perp}) = \tau_q^* \geq (\tau^*)^{\wedge} \geq (\tau_s(E))^{\wedge} = \tau_s(E/ F)$.

Proof: E/F is inductively semi reflexive i. e. $((E/F)'[\tau_q^*])' = E/F$ and so $\tau_k(E/F, F^{\perp}) = \tau_b(E/F, F^{\perp}) = \tau_q^*$. By lemma-2, $\tau_q^* \geq (\tau^*)^{\wedge}$. By [1], §22,2(1), $(\tau_s(E))_q = \tau_s(E/ F)$, and $(\tau^*)^{\wedge} \geq (\tau_s(E))$ is obvious.
Now we continue the proof of the theorem.

From lemma 3 and 4 we see that the sequence

$$o \rightarrow (F^{\perp}[(\tau^*)^{\wedge}]) \xrightarrow{q'} (E'[\tau^*]) \xrightarrow{i'} (F'[\tau^{\wedge *} = (\tau^*)_{q(E'/F^{\perp})}]) \rightarrow o \qquad (2).$$

is exact, algebraically and topologically.

Let E_1'' be the dual of $E'[\tau^*]$ i.e. $E_1'' = (E'[\tau^*])'$. Let $x'' \in E_1''$. The linear functional x'' is τ^*-continuous on E', its restriction $x''| F^{\perp}$ to the subspace $F^{\perp}=(E/F)'$ is $(\tau^*)^{\wedge}$-continuous and so it is in the dual of $(E/F)'[\tau_q^*]$.But E/F is inductively semi-reflexive, so $((E/F)'[\tau_q^*])' = E/F$. Hence $x''| F^{\perp} \in E/F$. So there exists $x_1 \in E$ such that $x''(u) = u(x_1)$ for all $u \in F^{\perp}$. Therefore, $x'' - x_1$ is a continuous linear functional on

$E'[\tau^*]$ which vanishes on $F^{\perp}$. In particular, for any neighborhood U of 0 in $E'[\tau^*]$, $x'' - x_1 \in U^o$. This means that $x'' - x_1$ is bounded linear functional on $U + F^{\perp}$. By lemma-1, $x'' - x_1$ is bounded on any neighborhood of 0 in $F'[\hat{\tau}^*]$. So there exists $x_2 \in F$ such that

$(x'' - x_1)(x') = x'(x_2)$ for all $x' \in F'$. It implies that $x'' = x_1 + x_2 \in E + F \subset E$. So $E_1'' = E$, that is, $(E'[\tau^*])' = E$. Hence $E[\tau]$ is inductively semi-reflexive.

Recall that a locally convex space $E[\tau]$ is called reinforced regular if every $\tau_s(E', (E'[\tau^*])')$-bounded set in the second dual $(E'[\tau^*])'$ is contained in $\tau_s(E', (E'[\tau^*])')$-closure of some bounded set in $E[\tau]$. Now we have the following, which is due to [16]:

Lemma-5: If the quotient map $q: E \rightarrow E/F$ lifts bounded sets with closure and if closed subspace F and the corresponding quotient E/F are reinforced regular, then E is reinforced regular.

Now we have:
2.10. Theorem([16]): If F is a closed subspace of a locally convex space $E[\tau]$ is such that both F and E/F are inductively reflexive, then $E[\tau]$ is inductively reflexive.

Proof: Given that the subspace F and the quotient E/F are inductively reflexive. Note that inductively reflexive space is always inductively semi-reflexive (by definition) and also B-semireflexive ([17]). We also note that B-semireflexive space is a complete reflexive space ([6]). So F and E/F are inductively semi-reflexive as well as barreled and complete. Now, firstly, by theorem- 2.9, the locally convex space $E[\tau]$ is inductively semi-reflexive. Secondly, three- space stability of barreledness (cf [14]) implies that $E[\tau]$ is barreled. Thirdly, three-space stability of completeness implies that $E[\tau]$ is complete. Now we prove that $E[\tau]$ is bornological (same as ultrabornological, since $E[\tau]$ is complete). We note that each topology ξ on the dual E' of $E[\tau]$ satisfying $\tau_c(E) \leq \xi \leq \tau_k(E)$ gives in E the same inductive topology ξ^* and this topology is not weaker than τ. Hence $E[\tau]$ is ultrabornological if and only if $\tau = \xi^*$.

Consider the sequences

$$o \rightarrow F \xrightarrow{i} E \xrightarrow{q} E/F \rightarrow o \qquad (1)$$

$$o \rightarrow (F^{\perp}[\tau_c(E)|\,F^{\perp}]) \xrightarrow{q'} (E'[\tau_c(E)]) \xrightarrow{i'} (F'[(\tau_c(E))_q) \rightarrow o \qquad (2)$$

The outer terms in (1) are inductively semi-reflexive and so reinforced regular. By the property of lifting of bounded sets by transposed mapping i′ and by lemma-5, the middle term E is reinforced regular. Hence $(\tau_c(E))^* = \tau_b(E') = \tau$ (see [5], proposition

3.2). Hence E[τ] is bornological. Thus E[τ] is inductively semi-reflexive and bornological and therefore, by [5], theorem1.7, E[τ] is inductively reflexive.

We investigate three-space stability of reflexivity in the following:
2.11. Theorem: Let E[τ] be an (F)-space and F a closed subspace of E[τ] such that F is also an (F)-space. If F and E/F are reflexive, then E[τ] is reflexive.

Proof: Assume that F and E/F are reflexive and so semi-reflexive. Since F as well as E is an (F)-space, therorem-2.6 is applicable. Therefore E[τ] is semi-reflexive. But an (F)-space is reflexive if and only if it is semi-reflexive (see [1]), so E[τ] is reflexive.
Recall that if E[τ] is B-semi-reflexive, then it is a complete reflexive space. **(**[6], theorem-12).

We, further, investigate three-space stability of B-semireflexivity and we have:
2.12. Theorem: Let E[τ] be an (F)-space and F a closed subspace of E[τ] such that F is also an (F)-space. If F and E/F are B-semireflexive, then E[τ] is B-semireflexive.

Proof: If F and E/F are B-semireflexive, then both F and E/F are reflexive and so by theorem-2.11, E[τ] is reflexive. Since E[τ] is an (F)-space, and a metrizable locally convex space is semi-reflexive if and only if it is B-semireflexive (see [17], theorem-2.5), hence E[τ] is B-semireflexive.

Bibliography

[1] G. Köthe:*Topological Vector Spaces I* ; (second printing revised) Springer-Verlag BerlinHeidelberg New York, 1983.

[2] KalmanBrauner: Duals of Frechet spaces and a Generalization of Banach-Dieudonné Theorem, Duke Math J. 40 (1973), 845-855.

[3] AtarsinghMeena and G.C. Dubey: Polar reflexivity in locally convex spaces, Global J. pure. Applied. Math. Vol.11, no.6(2015), 4349-4352.

[4] Seung-HyeokKye: Several Reflexivities in Topological Vector Spaces, J. Math. Anal. Appl. 139 (1989), 477-482.

[5] I. A. Berezanskií: Inductively Reflexive, Locally Convex Spaces, Sov. Math. Dokl.,9**,** No.5(1968),1080-1082; Translation from Dokl. Akad. NaukSSSR Tom182**,** no.1 (1968), 20-22.

[6] P. K. Raman: On a Class of Reflexive Spaces Related to Ulam's Conjecture on Measurable Cardinals, J. ReineAngew. Math. 245 (1970),188-200.

[7] A. C. Yorke: Geometry in quotient reflexive spaces, J. Austral. Math. Soc. (series A)35(1983), 263-268.

[8] James R. Clark: Coreflexive and somewhat reflexive Banach spaces: Proc. Amer. Math. Soc. Vol.36, No.1(1972), 421- 427.

[9] P. Civin and B. Yood: Quasi-reflexive spaces, Proc. Amer. Math. Soc.Vol.8(1957), 906-911.

[10] Kung-Wei Yang: A note on reflexive Banach spaces, Proc. Amer. Math. Soc. 18(1967), 859- 861.

[11] StozanRandenović, Some properties of C-reflexive locally convex spaces, Univ. Beograd Publ. Fak. Ser. Mat. 18 (2007), 52-58.

[12] StozanRadenović, Inheritance properties of some classes of locally convex spaces, Kragujevac J. Math, 29 (2006), 29-35.

[13] Steven F. Bellenot: An anti-open mapping theorem for Fréchet spaces, Pacific J. Math. Vol. 83, No.2 (1979), 325-331.

[14] W. Roelcke and S. Dierolf: *On three-space-problem for topological vector spaces;* Collect. Math. 32(1981), 13-35.

[15] Zoran Kadelburg ,StozanRadenović,; Three-space-problem for some classes of lineartopological spaces; Comment. Math. univ.Carolin,37,3 (1996), 507-514.

[16] StozanRadenović, Zoran Kadelburg: Three-space-problem for inductively(semi)-reflexivelocally convex spaces, PublcationsdeL'InstitutMathematique, Nouvelle série, tome 77,(91) (2005), 1-6.

[17] AtarsinghMeena and G.C. Dubey: Various reflexivities in locally convex spaces, International J. pure. Applied. Math.Sci. Vol.8, no.1(2015), 31-34.