International Journal of Mathematics Research.
ISSN 0976-5840 Volume 9, Number 1 (2017), pp. 5-12

http://www.irphouse.com

Certain Type of Pathway Fractional Integral Operator Associate via General Class of Polynomial

Hemlata Saxena[1] and Rajendra Kumar Saxena[2]

Professor of Mathematics[1] and Asst. Professor of Mathematics[2]
[1 & 2] Career Point University, Kota (Rajasthan) India.

Abstract

In the present paper we consider product of some special functions with a general class of polynomial via pathway fractional integral operator .This operator generalizes of the Riemann-Liouville fractional integral operator. Our results are quite general in nature .Some known and new results are also obtain here.

Keywords: Pathway fractional integral operator/ M-Series/ New generalized Mittag-Leffler function/ a general class of polynomials

INTRODUCTION

DEFINITION: Let $f(x) \in L(a,b)\,, \eta \in C\,, R(\eta) > 0, a > 0$ and let us take a "Pathway parameter" $\alpha < 1$.Then the pathway fractional integration operator is defined by Nair [8]

$$\left(P_{0^+}^{(\eta\,,\alpha)} f\right)(x) = x^{\eta} \int_0^{\left(\frac{x}{a(1-\alpha)}\right)} \left[1 - \frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{(1-\alpha)}} f(t)dt \qquad \text{......(1.1)}$$

when$\alpha = 0\,, a = 1\ and\ \eta$ is replaced by $\eta - 1$ in (1.1) it yields

$$\left(I_{0+}^{\eta} f\right)(x) = \frac{1}{\Gamma(\eta)} \int_0^x (x-t)^{\eta-1} f(t)dt \qquad \text{…(1.2)}$$

which is the left – sided Riemann-Liouville fractional integral defined by Samko et. al.[9].

The pathway model is introduced by Mathai [5] and studied further by Mathai and Houbold [6] [7].

For $R(\alpha) > 0$, the pathway model for scalar random variables is represented by the following probability density function.

$$f(x) = c|x|^{\gamma-1}\left[1 - a(1-\alpha)|x|^{\delta}\right]^{\frac{\beta}{1-\alpha}} \quad \text{.......(1.3)}$$

$$\gamma > 0, \delta > 0, \beta \geq 0, \{1 - a(1-\alpha)|x|^{\delta}\} > 0, -\infty < x < \infty,$$

where c is the normalizing constant and α is pathway parameter. For real, the normalizing constant is as follows:

$$c = \frac{1}{2}\,\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\,\Gamma(\frac{\gamma}{\delta} + \frac{\beta}{1-\alpha} + 1)}{\Gamma\left(\frac{\gamma}{\delta}\right)\,\Gamma(\frac{\beta}{1-\alpha} + 1)}\ , \qquad \alpha < 1 \quad \text{.......(1.4)}$$

$$= \frac{1}{2}\,\frac{\delta[a(1-\alpha)]^{\frac{\gamma}{\delta}}\,\Gamma(\frac{\beta}{1-\alpha})}{\Gamma\left(\frac{\gamma}{\delta}\right)\,\Gamma(\frac{\beta}{1-\alpha} - \frac{\gamma}{\delta})}\ , \qquad \text{for } \frac{1}{1-\alpha} - \frac{\gamma}{\delta} > 0, \alpha > 1 \quad \text{......(1.5)}$$

$$= \frac{1}{2}\,\frac{\delta[a\beta]^{\frac{\gamma}{\delta}}}{\Gamma\left(\frac{\gamma}{\delta}\right)}\ , \qquad \text{for } \alpha \to 1 \quad \text{......(1.6)}$$

For $\alpha < 1$, it is a finite range density with $\left[\,1 - a(1-\alpha)|x|^{\delta}\right] > 0$ and (1.3) remains in the extended generalized type-1 beta family. For $\alpha < 1$, the pathway density in (1.3) includes the extended type-1 beta density, the triangular density, the uniform density and many other p.d. f.

when $\alpha > 1$, we write $1 - \alpha = -(\alpha - 1)$, then

$$\left(P_{0^+}^{(\eta,\alpha)} f\right)(x) = x^{\eta}\int_0^{\left(-\frac{x}{a(\alpha-1)}\right)}\left[1 + \frac{a(\alpha-1)t}{x}\right]^{-\frac{\eta}{(\alpha-1)}} f(t)dt$$

$$f(x) = c\,|x|^{\gamma-1}\left[1 + a(\alpha-1)|x|^{\delta}\right]^{-\frac{\beta}{\alpha-1}} \quad \text{...... (1.7)}$$

Where $\alpha > 1, \delta > 0, \beta \geq 0, -\infty < x < \infty,$

which is extended generalized type-2 beta model for real x, It includes the type-2 beta density, the F density, the student-t density, the Cauchy density and many more.

Here, we consider only the case of pathway parameter $\alpha < 1$. For $\alpha \to 1$ both (1.3) and (1.7) take the exponential form, since.

$$\lim_{\alpha \to 1} c\,|x|^{\gamma-1}\left[1-a(1-\alpha)|x|^{\delta}\right]^{\frac{\eta}{1-\alpha}}$$

$$= \lim_{\alpha \to 1} c\,|x|^{\gamma-1}\left[1+a(\alpha-1)|x|^{\delta}\right]^{-\frac{\eta}{\alpha-1}}$$

$$= c|x|^{\gamma-1}e^{-a\eta|x|^{\delta}} \qquad \text{.........(1.8)}$$

For $\to 1_-$, $\left[1-\frac{a(1-\alpha)t}{x}\right]^{\frac{\eta}{1-\alpha}} \to e^{-\frac{a\eta}{x}t}$, the operator (1.1) reduces to the following form

$$\left(P_{0^+}^{(\eta,1)}f\right)(x) = x^{\eta}\int_0^{\infty} e^{-\frac{a\eta}{x}t}\, f(t)dt$$

$$= x^{\eta}L_f\left(\frac{a\eta}{x}\right) \qquad \text{......(1.9)}$$

It reduces to the Laplace integral transform of f with parameter $\frac{a\eta}{x}$.

In this paper we will integrate product of M-series, Fox's H-function and generalized Mittag-Leffler function by means of pathway model.

The following general class of polynomials introduced by Srivastava [13]

$$S_n^m[x] = \sum_{l=0}^{[n/m]} \frac{(-n)ml}{l!} A_{n,l}x^l$$

$$= \psi_1(l) \qquad l = 0,1,2 \qquad \text{............(1.10)}$$

When m is an arbitrary positive integer and the coefficients $A_{n,l}(n,l \geq 0)$ are arbitrary constants, real or complex

The generalized M-series is defined and studied by Sharma and Jain [11] as follows

$$_{\rho}M_{\sigma}^{\alpha',\beta'}(z) = \sum_{k=0}^{\infty} \frac{(a_1')_k \ldots\ldots (a_\rho')_k}{(b_1')_k \ldots\ldots (b_\sigma')_k} \frac{z^k}{\Gamma(\alpha' k+\beta')}$$

$$= \psi_1(k) \qquad \text{...(1.11)}$$

Where $z, \alpha', \beta' \in C, Re(\alpha') > 0$

Here $(a_j')_k, (b_j')_k$ are known as Pochammer symbols. The series (1.11) is defined when none of the parameters $b_{js}', j = 1,2, \ldots \sigma$ is negative integer or zero. The series in (1.11) is convergent for all z if $\rho \leq \sigma$, it is convergent for $|z| < \delta = \alpha^{\alpha}$ if $\rho = \sigma + 1$ and divergent, if $\rho > \sigma + 1$. When $\rho > \sigma + 1$ and $|z| < \delta$, the series can converge on conditions depending on the parameters.

The series representation of Fox H- function studied by Fox C [2] as follows:

$$H_{P,Q}^{M,N}\left[z\left|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right.\right]=\sum_{h=1}^{N}\sum_{v=0}^{\infty}\frac{(-1)^v X(\xi)}{v!\,E_h}\left(\frac{1}{z}\right)^{\xi}, \quad \dots(1.12)$$

where $\xi=\frac{e_h-v-1}{E_h}$ and $(h=1,2,\dots\ ,N)$

and

$$X(\xi)=\frac{\prod_{j=1}^{M}\Gamma(f_j+F_j\xi)\prod_{\substack{j=1\\j\neq h}}^{N}\Gamma(1-e_j+E_j\xi)}{\prod_{j=m+1}^{Q}\Gamma(1-f_j-F_j\xi)\prod_{j=N+1}^{P}\Gamma(e_j+E_j\xi)} \quad \dots..(1.13)$$

Following are the convergence conditions:

$$T_1=\sum_{i=1}^{N}E_i-\sum_{i=N+1}^{P}E_i+\sum_{i=1}^{M}F_i-\sum_{i=M+1}^{Q}F_i \quad \dots.(1.14)$$

$$T_2=\sum_{i=1}^{n}\alpha_i-\sum_{i=n+1}^{q}\alpha_i+\sum_{i=1}^{m}\beta_i-\sum_{i=m+1}^{q}\beta_i \quad \dots(1.15)$$

Recently, a new generalization of Mittag-Leffler function was defined by Faraj and Salim [3] as follows:

$$E_{\alpha,\beta,p}^{\gamma,\delta,q}(z)=\sum_{n=0}^{\infty}\frac{(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)\,(\delta)_{pn}} \quad \dots(1.16)$$

Where $z,\alpha,\beta,\gamma,\delta\in C;Min\{Re(\alpha),Re(\beta),Re(\gamma),Re(\delta)\}>0,p,q>0$

Further, generalization of Mittag- Leffler function was defined by Khan and Ahmed [4] as follows:

$$E_{\alpha,\beta,v,\sigma,\delta,p}^{\mu,\rho,\gamma,q}(z)=\sum_{n=0}^{\infty}\frac{(\mu)_{\rho n}(\gamma)_{qn}z^n}{\Gamma(\alpha n+\beta)(v)_{\sigma n}\,(\delta)_{pn}} \quad \dots(1.17)$$

Where $\alpha,\beta,\gamma,\delta,\mu,v,\rho,\sigma\in C;p,q>0$ and $q\leq Re(\alpha)+\rho p,$ and $Min\{Re(\alpha),$ $Re(\beta),Re(\gamma),Re(\delta),Re(\mu),Re(v),Re(\rho),Re(\sigma)\}>0$ If we take $\mu=v,\rho=\sigma$ in (1.17) it reduces to eq. (1.16).

Write generalized hypergeometric function was defined by Srivastava and Manocha [12] as follows:

$$ {}_p\psi_q\left[(a_1,A_1),\dots\left(a_p,A_p\right);(b_1,B_1),\dots,\left(b_q,B_q\right);z\right] $$

$$ = \sum_{n=0}^{\infty}\frac{\prod_{i=1}^{p}\Gamma(a_i+A_i n)z^n}{\prod_{j=1}^{q}\Gamma\left(b_j+B_j n\right)n!}\ S_{n'}^{M'}\left[\mathrm{d'}\mathrm{t}^{-\beta''}\right] \qquad \text{….(1.18)} $$

MAIN RESULTS

Theorem-1 Let $\eta,\gamma,\delta,q,p,\omega,\rho\in C,\ c,b\ \in R, Re(\beta)>0, Re(\delta)>0, Re(\eta)>0,$ $Re(\gamma)>0, Re(\omega)>0, Re\left(1+\frac{\eta}{1-\alpha}\right)>0, Re(\rho)>0, \alpha<1, b\in R, c\in R,$ $Re\left(\omega+\delta\frac{f_j}{F_j}\right)>0, |\arg c|<\frac{1}{2}T_1\pi, T_1T_2>0, \rho\le\sigma, |d|<\alpha'^{\alpha'}, \beta^*>0, j=1,\dots,Q;$

Then

$$ P_{0^+}^{(\eta,\alpha)}\left\{t^{\omega-1}\ {}_{\rho}^{\alpha',\beta'}M_{\sigma}\ \left[dt^{-\beta^*}\right].S_{n'}^{M'}\left[\mathrm{d'}\mathrm{t}^{-\beta''}\right]H_{P,Q}^{M,N}\left[ct^{\delta'}\left|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right.\right].E_{\alpha,\beta,p}^{\gamma,\delta,q}(bt^{\rho})\right\} $$

$$ =\psi_1(k)\,\psi_2(l)\frac{d^k[d']^l x^{\eta+\omega-\beta^*k}\,(\beta'')^l\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\Gamma(\delta)}{\Gamma(\gamma)[a(1-\alpha)]^{\omega-\beta^*k}-\ (\beta'')^l} $$

$$ {}_2\psi_3\left[\frac{bx^{\rho}}{[a(1-\alpha)]^{\rho}}\left|\begin{matrix}(\gamma,q)\ \ (\omega-\beta^*k,(\beta'')l-\delta\mathfrak{z},\rho)\\(\beta,\alpha)(\delta,\rho)\left(1+\omega+\frac{\eta}{1-\alpha}-\delta\xi-\beta^*k,(\beta'')l,\rho\right)\end{matrix}\right.\right]. $$

$$ H_{P,Q}^{M,N}\left[\frac{cx^{\delta'}}{[a(1-\alpha)]^{\delta'}}\left|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right.\right] \qquad \text{……….(2.1)} $$

Proof: The theorem -1 can be evaluated by using the definitions (1.1),(1.10),(1.11),(1.12) and (1.16) then by interchange the order of integrations and summations, evaluate the inner integral by making use of beta function formula, we arrive at the desired result (2.1).

Theorem-2 Let $\eta,\gamma,\delta,q,p,\beta,T_1,T_2>0,\mu,\rho,\gamma,\vartheta,\beta,v,\sigma,\delta\in C\ , Re(\eta)>0, Re(\gamma)$ $>0, Re(\beta)>0, Re\left(1+\frac{\eta}{1-\alpha}\right)>0, b,c\in R, \alpha<1, Re\left(\omega+\delta\frac{f_j}{F_j}\right)>0, |\arg c|<$ $\frac{1}{2}T_1\pi, \rho\le\sigma\ and\ |d|<\alpha'^{\alpha'}, \beta^*>0, j=1,\dots,Q$

$and\ \min(Re(\vartheta),Re(\beta),Re(\gamma),Re(\delta),Re(\mu),Re(v),Re(\rho),Re(\sigma))>0$

Then

$$P_{0^+}^{(\eta,\alpha)}\left\{t^{\beta-1}\ {}_{\rho}\overset{\alpha',\beta'}{M}{}_{\sigma}\ [dt^{-\beta^*}].S_{n'}^{M'}[\mathrm{d}'\mathrm{t}^{-\beta''}]\ H_{P,Q}^{M,N}\left[ct^{\delta'}\Big|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right].E_{\vartheta,\beta,v,\sigma,\delta,p}^{\mu,\rho,\gamma,q}\left(bt^{\vartheta}\right)\right\}$$

$$=\psi_1(k)\frac{d^k x^{\eta+\beta-\beta^*k-\beta''l}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)\Gamma(v)\Gamma(\delta)}{[a(1-\alpha)]^{\beta-\beta^*k-\beta''l}\Gamma(\mu)\Gamma(\gamma)}\,.\psi_2(l)\ (d')^l$$

$${}_3\psi_3\left[\frac{bx^{\vartheta}}{[a(1-\alpha)]^{\vartheta}}\middle|\begin{matrix}(\mu,\rho)\ \ (\gamma,q)\ \ (\beta-\beta^*k-\beta''l-\delta'\mathfrak{z},\vartheta)\\(\beta,\alpha)(\delta',\rho)(\vartheta,\sigma)\left(1+\beta+\frac{\eta}{1-\alpha}-\delta'\xi-\beta^*k-\beta''l,\vartheta\right)\end{matrix}\right].$$

$$H_{P,Q}^{M,N}\left[\frac{cx^{\delta'}}{[a(1-\alpha)]^{\delta'}}\middle|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right] \qquad ...(2.2)$$

Proof: The theorem -2 can be evaluated by using the definitions (1.1),(1.10) (1.11) and (1.16) then by interchange the order of integrations and summations, evaluate the inner integral by making use of beta function formula, we arrive at the desired result (2.2).

SPECIAL CASES:

1. If we take $\delta=p=q=1,\rho=\beta,\alpha=\beta,\beta=\omega$ and in H- function $\delta'=\delta$ in theorem -1 then we at once arrive at the known result of [1,Theorem-2].
2. If we take $\delta=p=1$ in theorem -1 then we get the following particular case of the solution (2.1)

Corollary-1

The following formula holds

$$P_{0^+}^{(\eta,\alpha)}\left\{t^{\omega-1}\ {}_{\rho}\overset{\alpha',\beta'}{M}{}_{\sigma}\ [dt^{-\beta^*}].\ H_{P,Q}^{M,N}\left[ct^{\delta'}\Big|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right].E_{\rho,\omega}^{\gamma,q}(bt^{\rho})\right\}$$

$$=\psi_1(k)\frac{d^k x^{\eta+\omega-\beta^*k}\,\Gamma\left(1+\frac{\eta}{1-\alpha}\right)}{\Gamma(\gamma)[a(1-\alpha)]^{\omega-\beta^*k}}$$

$${}_2\psi_2\left[\frac{bx^{\rho}}{[a(1-\alpha)]^{\rho}}\middle|\begin{matrix}(\omega-\delta\xi-\beta^*k,\rho)(\gamma,q)\\(\omega,\rho)\left(1+\omega+\frac{\eta}{1-\alpha}-\delta\xi-\beta^*k,\rho\right)\end{matrix}\right]$$

$$H_{P,Q}^{M,N}\left[\frac{cx^{\delta'}}{[a(1-\alpha)]^{\delta'}}\middle|\begin{matrix}(e_p,E_p)\\(f_Q,F_Q)\end{matrix}\right]$$

Where $\eta,\gamma,q,\omega,\rho \in C,\ c,b \in R, Re(\beta) > 0, Re(\delta) > 0, Re(\eta) > 0, Re(\gamma) > 0, Re(\omega) > 0, Re\left(1+\frac{\eta}{1-\alpha}\right) > 0, Re(\rho) > 0, \alpha < 1, b \in R, c \in R, Re\left(\omega+\delta\frac{f_j}{F_j}\right) > 0, |\arg c| < \frac{1}{2}T_1\pi, T_1T_2 > 0, \rho \leq \sigma, |d| < \alpha'^{\alpha'}, \beta^* > 0, j = 1, \dots, Q;$

3. If we take $\mu = v, \rho = \sigma, \delta = p = q = 1$ and $\vartheta \to \beta, \beta \to \omega$ in H function $\delta' = \delta$ in theorem-2 then we at once arrive at the known result of [1, Theorem-1] .
4. If we take $\mu = v, \rho = \sigma$ then we at once arrive at the theorem-1.
5. Making $\beta^*, \delta' \to 0$ and $\delta = p = q = 1, \rho = \beta$ in the result (2.1) and $\beta^*, \delta' \to 0, \mu = v, \rho = \sigma, \delta = p = q = 1$ in result (2.2) then we at once arrive at the known result of Nair in[8].
6. If we take $n' \to 0$ in theorem1 and theorem2 we get the result recently established by H. Saxena[10]
7. $\beta', \delta' and\ \ \eta' \to 0$ and $\delta = p = q = 1, \varphi = \beta\ in$ (2.1) and $\beta^*, \delta', \eta' \to 0$, $\mu = \vartheta\ \ \rho = \sigma, \delta = \rho = q = 1\ in$ (2.2) then we at once arrive at the know results of Nair in [8]

REFERENCES

[1] Chaurasia V. B. L. and Singh J., 2012, "Pathway Fractional Integral operator Associated with certain special functions", Global J. Sci. Front. Res., 12(9) (Ver.1.0).

[2] Fox, C., 1961, The G and H-functions as symmetrical Fourier kernels, Trans. Amer. Math soc., 98, pp. 395-429.

[3] Fraj, A, Salim, T., Sadek, S. and Ismail,J., 2013, " Generalized Mittag-Leffler function associated with Weyl Fractional Calculus Operators", J. of Mathematics Hindawi Publishing Corporation , ID 8217621,5 pages.

[4] Khan, M.A and Ahmed, S., 2013, "On some properties of the generalized Mittag-Leffler function", Springer plus, 2:337.

[5] Mathai, A.M., 2005, "A pathway to matrix variate gamma and normal densities", Linear Algebra and its Applications, 396, pp. 317-328.

[6] Mathai, A.M. and Haubold, H.J., 2008, "On generalized distribution and pathways", Phy. Letters, 372, pp. 2109-2113.

[7] Mathai, A.M. and Haubold, H.J., 2007, "Pathway models, superstatisties, trellis statistics and a generalized measure of entropy", physica, A 375, pp. 110-122.

[8] Nair, Seema S, 2009, "Pathway fractional integration operator", Fract. Cal.Appl. Anal., 12(3), 237-259.

[9] Samko, S.G., Kilbas, A.A. and Marichev, O.I., 1993, "Fractional integral and Derivatives". Theory and Applications, Gordenand Breach, Switzerland.

[10] Saxena H., Saxena, R.K., 2015, "On Certain type fractional integration of special functions via Pathway Operator", Global Journal Of Science Frontier Research (F) Vol.(15) issue 8 Version I .

[11] Sharma, Manoj and Jain, Renu, 2009, "A note on a generalized M-series as a special function of fractional calculus", Fract. Cal. Appl. Anal. 12(4), pp. 449-452.

[12] Srivastava, H.M. and Manocha, H.L., 1984, "A treatise on generating functions," John Wiley and Sons, Eillis Horwood, New York, Chichester.

[13] Srivastava, H.M., 1972, "A Contour integral involving Fox's H-function", Indian J-Math, 14, pp. 1-6.