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# pg**- compact spaces

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**Abstract**

The concepts of pg**-compact, pg**-countably compact, sequentially pg**-compact, pg**-locally compact and pg**- paracompact are introduced, and several properties are investigated. Also the concept of pg**-compact modulo *I* and pg**-countably compact modulo *I* spaces are introduced and the relation between these concepts are discussed.

**Keywords:** pg**-compact, pg**-countably compact, sequentially pg**-compact, pg**-locally compact, pg**- paracompact, pg**-compact modulo *I*, pg**-countably compact modulo *I*.

## 1. Introduction

Levine [3] introduced the class of g-closed sets in 1970. Veerakumar [7] introduced g*-closed sets. P M Helen[5] introduced g**-closed sets. A.S.Mashhour, M.E Abd El. Monsef [4] introduced a new class of pre-open sets in 1982. Ideal topological spaces have been first introduced by K.Kuratowski [2] in 1930. In this paper we introduce pg**-compact, pg**-countably compact, sequentially pg**-compact, pg**-locally compact, pg**-paracompact, pg**-compact modulo $I$ and pg**-countably compact modulo $I$ spaces and investigate their properties.

## 2. Preliminaries

Throughout this paper $(X, \tau)$ and $(Y, \sigma)$ represent non-empty topological spaces of which no separation axioms are assumed unless otherwise stated.

**Definition 2.1**

A subset $A$ of a topological space $(X, \tau)$ is called a pre-open set [4] if $A \subseteq int(cl(A)$ and a pre-closed set if $cl(int(A)) \subseteq A$.

**Definition 2.2** A subset $A$ of topological space $(X, \tau)$ is called

1. generalized closed set (g-closed) [3] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is open in $(X, \tau)$.
2. g*-closed set [7] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is g-open in $(X, \tau)$.
3. g**-closed set [5] if $cl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is g*-open in $(X, \tau)$.
4. pg**- closed set [6] if $pcl(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is g*-open in $(X, \tau)$.

**Definition 2.3** A function $f: (X, \tau) \rightarrow (Y, \sigma)$ is called

1. pg**-irresolute [6] if $f^{-1}(V)$ is a pg**-closed set of $(X, \tau)$ for every pg**-closed set $V$ of $(Y, \sigma)$.
2. pg**-continuous [6] if $f^{-1}(V)$ is a pg**-closed set of $(X, \tau)$ for every closed set $V$ of $(Y, \sigma)$.
3. pg**-resolute [6] if $f(U)$ is pg**- open in $Y$ whenever $U$ is pg**- open in $X$.

**Definition 2.4** An ideal [2] $I$ on a nonempty set $X$ is a collection of subsets of $X$ which satisfies the following properties. $(i)$ $A \in I$, $B \in I \Longrightarrow A \cup B \in I$ $(ii)$ $A \in I, B \subset A \Longrightarrow B \in I$. A topological space $(X, \tau)$ with an ideal $I$ on $X$ is called an ideal topological space and is denoted by $(X, \tau, I)$.

**Definition 2.5** [1] A collection $\mathcal{C}$ of subsets of $X$ is said to have finite intersection property if for every finite subcolection $\{C_1, C_2, \ldots, C_n\}$ of $\mathcal{C}$, the intersection $C_1 \cap C_2 \cap \ldots \cap C_n$ is nonempty.

## 3. pg**- Compact Space

We introduce the following definitions

**Definition 3.1** Let $X$ be a topological space. A collection$\{U_\alpha\}_{\alpha\in\Delta}$ of pg**-open subsets of $X$ is said to be *pg**-open cover* of $X$ if each point in $X$ belongs to at least one $U_\alpha$ that is, if $\bigcup_{\alpha\in\Delta} U_\alpha = X$.

**Definition 3.2** The topological space$(X,\tau)$ is said to be *pg**- compact* if every pg**-open covering $\mathcal{A}$ of $X$ contains a finite subcollection that also covers $X$. A subset $A$ of $X$ is said to be pg**-compact if every pg**-open covering of $A$ contains a finite subcollection that also covers$A$.

**Remark 3.3**

- A pg**- compact space is compact since every open set is pg**- open but not conversely.
- Any topological space having only finitely many points is necessarily pg**-compact, since in this case every pg**-open covering of $X$ is finite.

**Example 3.4** Let $X$ be an infinite set with cofinite topology. Then,

$$PG^{**}O(X) = \{\varphi, X, A\ /A^c is\ finite\}.$$

Let $\{U_\alpha\}_{\alpha\in\Delta}$ be an arbitrary pg**-open cover for $X$. Let $U_{\alpha_0}$ be a pg**- open set in the pg**- open cover $\{U_\alpha\}_{\alpha\in\Delta}$. Then $X - U_{\alpha_0}$ is finite say $\{ x_{1,} x_{2,} x_{3,} \dots, x_n \}$. Choose $U_{\alpha_i}$ such that $x_i \in U_{\alpha_i}$ for $i = 1,2,3,\dots,n$. Then $X = U_{\alpha_0} \cup U_{\alpha_1} \cup U_{\alpha_2} \cup \dots \cup U_{\alpha_n}$. Hence the space $X$ is pg**- compact and hence compact.

**Example 3.5** The real line $\mathbb{R}$ with usual topology is not pg**- compact since the pg**- open cover $\mathfrak{G} = \{(n, n+2)/n \in \mathbb{Z}\}$ has no finite subcover.

**Example 3.6** Let $X$ be an infinite indiscrete topological space. Obviously it is compact. But $\{\{x\}_{x\in X}\}$ is a pg**- open cover which has no finite sub cover. Hence it is not pg**- compact.

**Definition 3.7** A topological space $(\mathrm{X},\tau)$ is said to be *pg**-discrete* if every subset of $X$ is pg**-open. Equivalently every subset is pg**-closed.

**Theorem 3.8** A pg**-discrete space is pg**- compact if and only if the space is finite.

**Proof** Let $(X, \tau)$be pg**- discrete space. Suppose $X$ is a finite set, then obviously it is pg**- compact. Hence in particular every pg**- discrete finite space is pg**- compact. Conversely suppose $X$ is pg**- compact and assume that $X$ is an infinite set. Then the collection $\mathcal{A} = \{\{x\}: x \in X\}$ is a pg**- open cover of $X$. But $\mathcal{A}$ does not contain any finite subcover for $X$. Therefore $X$ is not pg**- compact, contrary to our supposition. Thus $X$ is a finite set.

**Theorem 3.9** Every pg**- closed subset of a pg**- compact space is pg**- compact but not conversely.

**Proof** Let A be a pg**- closed subset of the pg**- compact space $X$. Given a pg**- open covering $\{U_\alpha\}_{\alpha \in \Delta}$ of $A$. Let us form an pg**- open covering of $X$ by adjoining to $\{U_\alpha\}_{\alpha \in \Delta}$ the single pg**- open set $X - A$, that is $\{ \{U_\alpha\}_{\alpha \in \Delta} \cup (X - A) \}$. Some finite subcollection of $\{\{U_\alpha\}_{\alpha \in \Delta} \cup (X - A) \}$ covers $X$. If this subcollection contains the set $X - A$ , discard $X - A$, otherwise leave the subcollection alone. The resulting collection is a finite subcollection of $\{U_\alpha\}_{\alpha \in \Delta}$ that covers A.

**Example 3.10** Let $X = \{a, b, c\}, \tau = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$ . Here $PG^{**}O(X) = \{\varphi, \{a\}, \{b\}, \{a, b\}, X\}$. $X$ is pg**- compact. $Y = \{a, b\}$ is pg**- compact but not pg**- closed.

**Theorem 3.11** Let $X$ and $Y$ be topological spaces and $f : (X, \tau) \to (Y, \sigma)$ be a function. Then,

1. $f$ is pg**- irresolute and $A$ is a pg**- compact subset of $X \Longrightarrow f(A)$ is a pg**- compact subset of $Y$.
2. $f$ is one-one pg**-resolute map and $B$ is a pg**- compact subset of Y $\Longrightarrow f^{-1}(B)$ is a pg**-compact subset of $X$.

**Proof** (1) & (2) obvious from the definitions.

The following theorems give several equivalent forms of pg**-compactness which are often easier to apply.

**Theorem 3.12** If $A_1, A_2, A_3, \dots, A_n$ are pg**-compact subsets of a pg**- multiplicative space $X$ then $A_1 \cup A_2 \cup A_3 \cup \dots \cup A_n$ is also pg**-compact.

**Proof** Let $A = \overset{n}{\underset{i=1}{\cup}} A_i$. Suppose $\mathcal{A} = \{U_\alpha\}_{\alpha\in\Delta}$ is an pg**-open cover for $A$. Then $\mathcal{A}$ is a pg**-open cover for $A_1$, $A_2$, $A_3$, ... , $A_n$ separately also. Since each $A_i$ is pg**-compact there exists finite subcovers $\mathcal{A}_i$'s of each $A_i$, then $\overset{n}{\underset{i=1}{\cup}} \mathcal{A}_i$ forms a finite subcover of $\overset{n}{\underset{i=1}{\cup}} A_i = A$. Therefore $A$ is pg**-compact.

**Theorem 3.13** A topological space is pg**-compact if and only if every collection of pg**-closed sets with empty intersection has a finite sub collection with empty intersection.

**Proof** Follows from the fact that a collection of pg**-open sets is a pg**-open cover if and only if the collection of all their complements has empty intersection.

**Theorem 3.14** A topological space $X$ is pg**-compact ⇔ for every collection $\mathcal{C}$ of pg**-closed sets in X having the finite intersection property, the intersection $\underset{c\in\mathcal{C}}{\cap} C$ of all the elements of $\mathcal{C}$ is nonempty.

**Proof** Let $(X,\tau)$be pg**-compact and $\mathcal{C}$ be a collection of pg**-closed sets with finite intersection property. Suppose $\underset{c\in\mathcal{C}}{\cap} C = \varphi$ then $\{X - C\}_{C\in\mathcal{C}}$ is a pg**-open cover for $X$. Therefore there exists $C_1$ , $C_2$ , $C_3$ , ..., $C_n \in \mathcal{C}$ such that $\overset{n}{\underset{i=1}{\cup}} (X - C_i) = X$. This implies $\overset{n}{\underset{i=1}{\cap}} C_i = \varphi$ which is a contradiction. ∴ $\underset{c\in\mathcal{C}}{\cap} C \neq \varphi$. Conversely, assume the hypothesis given in the statement. To prove $X$ is pg**-compact. Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a pg**-open cover for $X$. Then $\underset{\alpha\in\Delta}{\cup} U_\alpha = X \Longrightarrow \underset{\alpha\in\Delta}{\cap} (X - U_\alpha) = \varphi$. By the hypothesis $\alpha_1$ , $\alpha_2$ , $\alpha_3$ , ... , $\alpha_n$ such that $\overset{n}{\underset{i=1}{\cap}} (X - U_{\alpha_i}) = \varphi$. Therefore $\overset{n}{\underset{i=1}{\cup}} U_{\alpha_i} = X$ and hence $X$ is pg**-compact.

**Corollary 3.15** Let $(X,\tau)$be pg**-compact space and let $C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$ be a nested sequence of non-empty pg**-closed sets in X. Then the intersection $\underset{n\in Z_+}{\cap} C_n$is nonempty.

**Proof** Obviously $\{C_n\}_{n\in Z_+}$has finite intersection property. Therefore by the previous theorem $\underset{n\in Z_+}{\cap} C_n$is nonempty.

**Theorem 3.16** Let $(X, \tau)$ and $(Y, \sigma)$ be two topological spaces and $f : (X, \tau) \rightarrow (Y, \sigma)$ be a function. Then,

1. $f$ is onto, pg**- continuous and $X$ is pg**- compact $\Longrightarrow Y$ is compact.
2. $f$ is onto, continuous and $X$ is pg**- compact $\Longrightarrow Y$ is compact.
3. $f$ is strongly pg**- irresolute and $X$ is compact $\Longrightarrow Y$ is pg**- compact.
4. $f$ is bijection and pg**- resolute then $Y$ is pg**- compact $\Longrightarrow X$ is compact.
5. $f$ is a bijection and pg**- open then $Y$ is pg**- compact $\Longrightarrow X$ is compact.
6. $f$ is onto, pg**- irresolute and $X$ is pg**- compact $\Longrightarrow Y$ is pg**- compact.
7. $f$ is a bijection and pg**- resolute then $Y$ is pg**- compact $\Longrightarrow X$ is pg**- compact.

**Proof** (1) Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a pg**-open cover for $Y$. Then $\{f^{-1}(U_\alpha)\}_{\alpha\in\Delta}$ is a pg**-open cover for $X$. Since $X$ is pg**-compact, there exists $\alpha_1$, $\alpha_2$, $\alpha_3, \dots, \alpha_n$ such that $X \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$. Then $Y = f(X) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Therefore $Y$ is compact.
Proofs for (2) to (7) are similar to the above.

**Remark 3.17** The property of being pg**- compact, is a pg**- topological property. This follows from (6) and (7) of the above theorem.

**Definition 3.18** A sequence $\langle x_n \rangle$ in topological space$(X, \tau)$ is *pg**- congregates* to $x$ in $X$ ($\langle x_n \rangle \xrightarrow{pg^{**}} x$) is for every pg**-neighbourhood $U$ of $x$ there exists a positive integer $N$ such that $x_n \in U$, $\forall\, n \geq N$. We say that $x$ is the *pg**-limit* of the sequence $\langle x_n \rangle$.

**Result 3.19** In a topological space $(X, \tau)$, if a sequence $\langle x_n \rangle$ *pg**- congregates* to $x_0$ then the set $A = \{x_0, x_1, x_2, \dots\}$ is pg**- compact in X.

**Proof** Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a pg**-open cover $\mathcal{A}$ for $A$, then $x_0 \in U_{\alpha_0}$for some $\alpha_0 \in \Delta$. Since $\langle x_n \rangle \xrightarrow{pg^{**}} x$, there exists a positive integer $N$ such that $x_n \in U_{\alpha_0}$, $\forall\, n \geq N$. Therefore the finite set $\{x_0, x_1, \dots, x_{N-1}\}$ can be covered by atmost a finite number of pg**-open sets $U_1, U_{2,} \dots, U_{N-1}$. Therefore the pg**-open cover $\mathcal{A}$ contains a finite subcover $\mathcal{C} = \{U_{\alpha_0}, U_1, U_{2,} \dots, U_{N-1}\}$ of $A$. Hence $A$ pg**- compact in X.

**Theorem 3.20** Let $(X, \tau)$ be a pg**- compact pg **- multiplicative space then every infinite subset of X has a pg**-cluster point.

**Proof** Let $(\mathrm{X}, \tau)$ be a pg**- compact pg**- multiplicative space and $A$ be a subset of X. We desire to prove that, if $A$ is infinite then $A$ has a pg**-cluster point. We prove it contrapositively, that is 'if $A$ has no pg**-cluster point, then $A$ must be finite'. Suppose $A$ has no pg**-cluster point, then $A$ is pg**-closed. Furthermore, for each $a \in A$ we can choose a pg**-neighbourhood $U_a$ of $a$ such that $U_a \cap A = \{a\}$. Then the space X can be covered by $\{(X - A), \{U_a\}_{a \in A}\}$, being pg**- compact, it can be covered by finitely many of these sets. Since $(X - A)$ does not intersect $A$ and each $U_a$ contains only one point of $A$, the set $A$ must be finite.

**Theorem 3.21 (Generalization of extreme value theorem):** Let $f : X \to Y$ be pg**- continuous, where $Y$ is an ordered set in the order topology. If $X$ is pg**-compact, then there exist points $c$ and $d$ in $X$ such that $f(c) \leq f(x) \leq f(d)\ \forall\ x \in X$.

**Proof** Since $f$ is pg**- continuous and $X$ is pg**- compact, The set $A = f(X)$ is compact. Suppose A has no largest element then the collection $\{(-\infty, a)/a \in A\}$ forms an open covering of A. Since A is compact it has some finite subcover $\{(-\infty, a_1), (-\infty, a_2), \dots, (-\infty, a_n)\}$. Let $a$ = max $\{a_1, a_2, a_3, \dots, a_n\}$, then $a$ belong to none of these sets, contrary to the fact that they cover $A$. $\therefore A$ has a largest element $M$. Similarly it can be proved that it has the smallest element m. Therefore there exists c and d in $X$ such that $f(c) = m,\ f(d) = M$ and $f(c) \leq f(x) \leq f(d)\ \forall\ x \in X$.

**Definition 3.22** A family $\mathcal{C}$ of subsets of a topological space $X$ is said to be *pg**-short* if $\mathcal{C}$ is not a pg**-open cover of $X$. $\mathcal{C}$ is said to be *finitely pg**-short* if no finite subcollection of pg**-open covering $\mathcal{C}$ covers $X$.

**Theorem** A topological space $(\mathrm{X}, \tau)$ is pg**- compact if and only if each finitely pg**-short family of pg**-open sets in $X$ is pg**-short.

**Proof** Suppose $X$ is pg**- compact. Let $\mathcal{C}$ be any finitely pg**-short family of pg**-open sets in $X$, then no finite subcollection of $\mathcal{C}$ covers $X$. We show that $\mathcal{C}$ does not cover $X$. Suppose $\mathcal{C}$ is a pg**-open cover of $X$. Then $\mathcal{C}$ is a pg**-open cover of $X$ which has no finite subcover which is a contradiction to $X$ is pg**- compact. Therefore $\mathcal{C}$ is pg**-short. Conversely assume each finitely pg**-short family of pg**-open sets in $X$ is pg**-short. Suppose $X$ is not pg**- compact, then there exist a pg**-open cover $\mathcal{A}$ of $X$ which has no finite subcover. Therefore $\mathcal{A}$ is a finitely pg**-short family of pg**-open sets in $X$, then by hypothesis, $\mathcal{A}$ is pg**-short that is $\mathcal{A}$ does not cover $X$, which is a contradiction. Hence $X$ is pg**- compact.

**Definition 3.23** A function $f: X \to Y$ between topological spaces is said to be *pg**-proper* if $f^{-1}(C)$ is pg**- compact for each pg**- compact subset $C$ of $Y$.

**Definition 3.24** A function $f: (X, \tau) \to (Y, \sigma)$ is called pg**- resolve if $f(F)$ is pg**- closed in $Y$ whenever $F$ is pg**- closed in $X$.

**Theorem 3.25** If $f: X \to Y$ is a pg**- resolve map between pg**- multiplicative spaces $X$ and $Y$. Also $f^{-1}(y)$ is pg**- compact for each $y \in Y$, then $f$ is pg**-proper.

**Proof** Let $C \subset Y$ be pg**- compact and let $\{U_\alpha / \alpha \in A\}$ be a collection of pg**-open sets whose union contains $f^{-1}(C)$. For any $y \in C$ there is a finite subset $A_y \subset A$ such that $f^{-1}(y) \subset \bigcup\{U_\alpha / \alpha \in A_y\}$ . Take $W_y = \bigcup\{U_\alpha / \alpha \in A_y\}$ and $V_y = Y - f(X - W_y)$, which is pg**-open. Now $f^{-1}(V_y) \subset W_y$ and $y \in V_y$. Since $C$ is pg**- compact and is covered by $V_y$, there are points $y_1, y_2, \ldots, y_n$ such that $C \subset V_{y_1} \cup V_{y_2} \cup \ldots \cup V_{y_n}$. Thus $f^{-1}(C) \subset f^{-1}(V_{y_1}) \cup f^{-1}(V_{y_2}) \cup \ldots \cup f^{-1}(V_{y_n}) \subset W_{y_1} \cup W_{y_2} \cup \ldots \cup W_{y_n}$. This implies $f^{-1}(C) \subset \bigcup\{U_\alpha / \alpha \in A_{y_i}; i = 1,2, \ldots, n\}$ which is finite. Therefore $f^{-1}(C)$ is pg**- compact and hence $f$ is pg**-proper.

## 4. pg**- compact modulo *I*

**Definition 4.1** An ideal topological space $(X, \tau, I)$ is said to be *pg**- compact modulo I* if for every pg**-open covering $\{U_\alpha\}_{\alpha \in \Delta}$ of X, there exists a finite subset $\Delta_0$ of $\Delta$ such that $X - \bigcup_{\alpha \in \Delta_0} U_\alpha \in I$.

**Remark 4.2** pg**- compactness implies pg**- compact modulo *I* for any ideal *I* but not conversely.

**Example 4.3** Let $(X, \tau, I)$ be an infinite indiscrete topological space where $I = \wp(X)$. Let $\{U_\alpha\}_{\alpha \in \Delta}$ be a pg**-open cover for $X$. Let $\alpha_0 \in \Delta$ then, $X - U_{\alpha_0} \in I$. Therefore $(X, \tau, I)$ is pg**- compact modulo $I$ but not pg**- compact.

**Note** "pg**- compact modulo *I*" and "pg**- compact" happens together when $I = \{\varphi\}$.

**Remark 4.4** pg**- compact modulo $I$ implies compact modulo *I* for any ideal *I* but not conversely.
Proof is obvious, since $\tau \subseteq G^{**}O(X)$.

**Example 4.5** An indiscrete space $(X, \tau, \{\varphi\})$ is compact modulo $I$ but not pg**-compact modulo $I$.

**Theorem 4.6** Let X be an ideal topological space. Then the following are equivalent.

1) X is pg**- compact modulo $I$.
2) For every family $\{F_\alpha / \alpha \in \Omega\}$ of pg**-closed sets such that $\bigcap_{\alpha\in\Omega} F_\alpha = \varphi$, then there exists a finite sub family $\{F_{\alpha_i}\}_{i=1}^{n}$ such that $\bigcap_{i=1}^{n} F_{\alpha_i} \in I$.
3) For every family $\{F_\alpha / \alpha \in \Omega\}$ of pg**-closed sets with $I$-FIP $\bigcap_{\alpha\in\Omega} F_\alpha \neq \varphi$

**Proof**

$(1) \Rightarrow (2)$ Let X be pg**- compact modulo $I$ and $\{F_\alpha / \alpha \in \Omega\}$ be a family of pg**-closed sets such that $\bigcap_{\alpha\in\Omega} F_\alpha = \varphi$. Therefore $\cup F_\alpha^c = X$ where $F_\alpha^c$ is pg**-open. Hence $\{F_\alpha^c / \alpha \in \Omega\}$ is a pg**-open cover for X, also since X is pg**- compact modulo $I$ there exists $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ such that $X - \bigcup_{i=1}^{n} F_{\alpha_i}^c \in I$. This implies $\bigcap_{i=1}^{n} F_{\alpha_i} \in I$.

$(2) \Rightarrow (3)$ Let $\{F_\alpha / \alpha \in \Omega\}$ be a family of pg**-closed sets with $I$-FIP. Suppose $\bigcap_{\alpha\in\Omega} F_\alpha = \varphi$, then there exists $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ such that $\bigcap_{i=1}^{n} F_{\alpha_i} \in I$. Which contradicts the hypothesis. Therefore $\bigcap_{\alpha\in\Omega} F_\alpha \neq \varphi$.

$(3) \Rightarrow (1)$ Let $\{U_\alpha / \alpha \in \Omega\}$ be a pg**-open cover for X. Then $\bigcap_{\alpha\in\Omega} U_\alpha^c = \varphi$. Therefore the family of pg**-closed sets $\{U_\alpha^c / \alpha \in \Omega\}$ does not satisfy $I$-FIP and therefore there exists $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ such that $\bigcap_{i=1}^{n} U_{\alpha_i}^c \in I$ (ie) $X - \bigcup_{i=1}^{n} U_{\alpha_i} \in I$. Therefore X is pg**-compact modulo $I$.

**Theorem 4.7** If $I \subseteq J$, then $(X, \tau, I)$ is pg**- compact modulo $I$ implies $(X, \tau, J)$ is pg**-compact modulo $J$.
Proof is obvious.

**Theorem 4.8** Let $I_F$ denote the ideal of all finite subsets of $X$. Then $(X, \tau)$ is pg**-compact if and only if $(X, \tau, I_F)$ is pg**- compact modulo $I_F$.

**Proof**

Necessity follows since $\{\varphi\} \in I_F$.

Sufficiency Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a pg**-open cover for $X$, then there exists a finite subset $\Delta_0$ of $\Delta$ such that $X - \bigcup_{\alpha\in\Delta_0} U_\alpha \in I_F$. $X - \bigcup_{\alpha\in\Delta_0} U_\alpha = \{x_1, x_2, x_3, \dots, x_n\}$. Choose $\alpha_i$ such that $x_i \in U_{\alpha_i}$ for $i = 1, 2, 3, \dots, n$. Then $X = \{\bigcup_{\alpha\in\Delta_0} U_\alpha\} \cup \left\{\bigcup_{i=1}^{n} U_{\alpha_i}\right\}$. Therefore $X$ is pg**- compact.

## 5. pg**- countably compact space

**Definition 5.1** A subset $A$ of a topological space$(\mathrm{X}, \tau)$ is said to be *pg**- countably compact* if every countable pg**-open covering of $A$ has a finite sub cover.

**Example 5.2** An infinite. cofinite topological space is pg**- countably compact.

**Example 5.3** A countably infinite indiscrete topological space is not pg**- countably compact.

**Remark 5.4** Every pg**- compact space is pg**- countably compact.

**Theorem 5.5** Every pg**- closed subset of a pg**- countably compact space is pg**- countably compact.
Proof is similar to theorem (3.9)

**Theorem 5.6** A topological space $X$ is pg**- countably compact $\Leftrightarrow$ for every collection $\mathcal{C}$ of pg**-closed sets in $X$ having the finite intersection property, the intersection $\bigcap_{c\in\mathcal{C}} C$ of all the elements of $\mathcal{C}$ is nonempty.
Proof is similar to the proof of theorem (3.14)

**Corollary 5.7** Let $(X, \tau)$ be pg**- countably compact space and let $C_1 \supset C_2 \supset C_3 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$ be a nested sequence of non-empty pg**-closed sets in $X$. Then the intersection $\bigcap_{n\in Z_+} C_n$ is nonempty.

**Proof** Obviously $\{C_n\}_{n\in Z_+}$has finite intersection property. Therefore by theorem (5.6) $\bigcap_{n\in Z_+} C_n$is nonempty.

**Theorem 5.8** A topological space $(X, \tau)$ is pg**-countably compact if and only if every infinite subset has a pg**-cluster point.

**Proof** Suppose if every infinite subset of $(X,\tau)$ has a pg**-cluster point and $\mathfrak{F} = F_i$ countable collection of pg**-closed sets with finite intersection property. The intersection $H_n = \bigcap_{j=1}^{n} F_j$ is nonempty for all $n$. Choose $x_n \in H_n$ for each $n$. Then the set $E = \{x_n \,/\, x_n \in H_n\}$ has a pg**-cluster point $x$ . But $x_n \in F_i$ for all $n \geq i$, since $F_i$ is pg**-closed $x \in F_i$ , $\forall\, i$. Therefore $x \in \cap\, F_i$, hence every countable collection $\mathfrak{F}$ of pg**-closed sets with finite intersection property has a nonempty intersection, then by theorem (5.6) $X$ is pg**- countably compact.
Conversely let $(X,\tau)$be pg**- countably compact and suppose that there exists an infinite subset $A$ has no pg**-cluster point. Let $E = \{x_n \,/ n \in N\}$ be a countable subset of $A$ . Since $E$ has no pg**-cluster point of $E$ , there exists a pg**-neighbourhood $U_n$ of $x_n$ such that $E \cap U_n = \{x_n\}$. Therefore $\{E^c, \{U_n\}_{n\in N}\}$ is a countable pg**-open cover for $X$. This pg**-open cover has no finite subcover, this is a contradiction for $X$ is pg**- countably compact. Therefore every infinite subset of $X$ has a pg**-cluster point.

**Theorem 5.9** Let $(\mathrm{X},\tau)$ and $(\mathrm{Y},\sigma)$ be two topological spaces and $f : (\mathrm{X},\tau) \to (\mathrm{Y},\sigma)$ be a bijective function. Then,

1. $f$ is pg**- continuous and $X$ is pg**- countably compact $\Longrightarrow$ $Y$ is countably compact.
2. $f$ is continuous and $X$ is pg**- countably compact $\Longrightarrow$ $Y$ is countably compact.
3. $f$ is strongly pg**- irresolute and $X$ is countably compact $\Longrightarrow$ $Y$ is pg**- countably compact.
4. $f$ is pg**- resolute and $Y$ is pg**- countably compact $\Longrightarrow$ $X$ is countably compact.
5. $f$ is pg**-open and $Y$ is pg**- countably compact $\Longrightarrow$ $X$ is countably compact.
6. $f$ is pg**- irresolute and $X$ is pg**- countably compact $\Longrightarrow$ $Y$ is pg**- countably compact.
7. $f$ is pg**- resolute and $Y$ is pg**- countably compact $\Longrightarrow$ $X$ is pg**- countably compact.

**Proof** Let $\{U_\alpha\}_{\alpha\in\Delta}$ be a pg**- countable open cover for $Y$. Then $\{f^{-1}(U_\alpha)\}_{\alpha\in\Delta}$ is a pg**- countable open cover for $X$. Since $X$ is pg**- countably compact, there exists $\alpha_1$ , $\alpha_2$ , $\alpha_3$ , ... , $\alpha_n$ such that $X \subseteq \bigcup_{i=1}^{n} f^{-1}(U_{\alpha_i})$. Then $Y = f(X) \subseteq \bigcup_{i=1}^{n} U_{\alpha_i}$. Therefore $Y$ is countably compact.
Proofs for (2) to (7) are similar to the above.

**Remark 5.10** The property of being pg**- countably compact, is a pg**- topological property. This follows from (6) and (7) of the above theorem.

## 6. pg**- countably compact modulo *I*

**Definition 6.1** An ideal topological space $(X, \tau, I)$ is said to be *pg**-countably compact modulo I* if for every countable pg**-open covering $\{U_\alpha\}_{\alpha\in\Delta}$ of $X$, there exists a finite subset $\Delta_0$ of $\Delta$ such that $X - \bigcup_{\alpha\in\Delta_0} U_\alpha \in I$.

All the results from remark (4.2) to theorem are true in the case when $(X, \tau, I)$ is pg**-countably compact modulo *I*.

## 7. Sequentially pg**- compact

**Definition 7.1** A subset $A$ of a topological space $(X, \tau)$ is said to be *sequentially pg**-compact* if every sequence in $A$ contains a subsequence which pg**- congregates to some point in $A$.

**Example 7.2** A finite topological space is sequentially pg**- compact.
Let $\langle x_n \rangle$ be an arbitrary sequence in a finite topological space $X$. Since $X$ is finite $x_n = x_0$ except for finitely many $n$'s, therefore the constant sequence $x_0, x_0, \dots$ is pg**- congregates to $x_0$ in $X$. Hence $X$ is sequentially pg**- compact.

**Example 7.3** An infinite indiscrete topological space is not sequentially pg**- compact.
Let $(X, \tau)$ be an infinite indiscrete topological space and let $x_0 \in X$ be arbitrary. Since all the subsets are pg**-open in this space $\{x_0\}$ is a pg**-neighbourhood of $x_0$, therefore no sequence other than the constant sequence $\langle x_0 \rangle$ can pg**- congregates to $x_0$ in $X$. Since $x_0$ is arbitrary $(X, \tau)$ is not sequentially pg**- compact.

**Remark 7.4** Sequentially pg**- compactness implies sequentially compactness, but the reverse implication is not true as seen in the following example.

**Example 7.5** Every infinite indiscrete topological space is sequentially compact but not sequentially pg**- compact.

**Theorem 7.6** A finite subset $A$ of a topological space $(X, \tau)$ is sequentially pg**- compact.

**Proof** Let $\langle x_n \rangle$ be an arbitrary sequence in a finite subset $A$. At least one element of the sequence say $x_0$ must be repeated infinite number times, since $A$ is finite. Hence the constant subsequence $x_0, x_0, \ldots$ is pg**- congregates to $x_0$ in $X$.

**Theorem 7.7** If a pg**- congregate sequence in a topological space has infinitely many distinct points, then its pg**-limit is a pg**-limit point of the set of points of the sequence.

**Proof** Let $(\mathrm{X}, \tau)$ be a topological space and let $\langle x_n \rangle$ be a pg**- congregate sequence in $X$ with pg**-limit $x$. Assume that $x$ is not a pg**-limit point of the set $A$ of points of the sequence, and show that the sequence has only finitely many distinct points. Since $x$ is not a pg**-limit point of $A$ there exists a pg**-neighbourhood $U$ of $x$ contains no point of the sequence different from $x$. However, since $x$ is the pg**-limit of the sequence, all $x_n$'s must lie in $U$, hence must coincide with $x$. Therefore there are only finitely many distinct points in the sequence.

**Theorem 7.8** A topological space $(X, \tau)$ is sequentially pg**- compact if and only if every infinite subset has a pg**-limit point.

**Proof** Assume that $X$ is sequentially pg**- compact and let $A$ be an infinite subset of X. Since $A$ is infinite a sequence $\langle x_n \rangle$ of distinct points can be extracted from $A$. Since X is sequentially pg**- compact this sequence has a subsequence which pg**- congregates to a point $x$. Hence from theorem (7.7) $x$ is a pg**-limit point of the set of points of the subsequence, since this set is a subset of $A$, $x$ is also a pg**-limit point of $A$. Conversely assume that every infinite subset of X has a pg**-limit point. Let $\langle x_n \rangle$ be an arbitrary sequence in X. If $\langle x_n \rangle$ has a point which is infinitely repeated, then it has a constant subsequence and this subsequence is pg**- congregate. If no point of $\langle x_n \rangle$ is infinitely repeated, then the set $A$ of points of this sequence is infinite, then we can extract the subsequence from $\langle x_n \rangle$ which is pg**- congregate to $x$ since the set $A$ has a pg**-limit point $x$.

**Theorem 7.9** Let $(\mathrm{X}, \tau)$ and $(\mathrm{Y}, \sigma)$ be two topological spaces, then X and Y are sequentially pg**- compact if and only if X × Y is sequentially pg**- compact.

**Proof** Suppose that X and Y are sequentially pg**- compact. If $(x_n, y_n)$ is a sequence in X × Y,then there exists a subsequence $(x_{n_k})$ of $(x_n)$ such that $(x_{n_k}) \xrightarrow{pg^{**}} a$ in X.

Similarly there is a subsequence $(y_{n_k})$ of $(y_n)$ such that $(y_{n_k}) \xrightarrow{pg^{**}} b$ in Y. Thus $(x_{n_k}, y_{n_k}) \xrightarrow{pg^{**}} (a, b)$. Therefore X × Y is sequentially pg**- compact. To prove the converse consider the projection maps $\pi_x$: X × Y → X defined by $\pi_x(x, y) = x$ and $\pi_y$: X × Y → Y defined by $\pi_y(x, y) = y$. Suppose X × Y is sequentially pg**- compact. If $(x_n)$ and $(y_n)$ are sequences of X and Y respectively then $(x_n, y_n)$ is a sequence in X × Y. Then there exists a subsequence $(x_{n_k}, y_{n_k})$ of $(x_n, y_n)$ such that $(x_{n_k}, y_{n_k}) \xrightarrow{pg^{**}} (a, b)$ since X × Y is sequentially pg**- compact. This implies $(x_{n_k}) \xrightarrow{pg^{**}} a$ in X and $(y_{n_k}) \xrightarrow{pg^{**}} b$ in Y. Therefore X and Y are sequentially pg**- compact.

**Theorem 7.10** Let (X, τ) and (Y, σ) be two topological spaces and $f$ : (X, τ) → (Y, σ) be a function. Then,

1. $f$ is a bijection and pg**- resolute then Y is sequentially pg**- compact ⟹ $X$ is sequentially pg**- compact.
2. $f$ is onto, pg**- irresolute and $X$ is sequentially pg**- compact ⟹ $Y$ is sequentially pg**-compact.
3. $f$ is onto, strongly pg**- irresolute and $X$ is sequentially compact ⟹ Y is sequentially pg**- compact.
4. $f$ is onto, continuous and $X$ is sequentially pg**- compact ⟹ Y is sequentially compact.

**Proof** (1) If $\langle x_n \rangle$ is a sequence in $X$, then $\langle f(x_n) \rangle$ is a sequence in $Y$. Since Y is sequentially pg**- compact, $\langle f(x_n) \rangle$ has a pg**- congregate subsequence $\langle f(x_{n_k}) \rangle$ such that $f(x_{n_k}) \xrightarrow{pg^{**}} y_0$ in Y. Then there exists $x_0 \in X$ such that $f(x_0) = y_0$. Let $U$ be a pg**-open set containing $x_0$, then $f(U)$ is a pg**-open set containing$f(x_0)$. Then there exists $N$ such that $f(x_{n_k}) \in f(U)$, $\forall\, k \geq N$. Therefore $X$ is sequentially pg**- compact.

Proofs for (2) to (4) are similar to the above.

**Remark 7.11** The property of being sequentially pg**-compact, is a pg**- topological property. This follows from (1) and (2) of the above theorem.

**Theorem 7.12** Every sequentially pg**-compact is pg**- countably compact.

**Proof** Let $(X,\tau)$ be sequentially pg**-compact. Suppose X is not pg**- countably compact, then there exists countable pg**-open covering $\{U_\alpha\}_{\alpha\in\Delta}$such that $\underset{\alpha\in\Delta}{\cup} U_\alpha = X$ , which has no finite subcover. Choose $x_1 \in U_1, x_2 \in U_2 - U_1, x_3 \in U_3 - \underset{i=1,2}{\cup} U_i \ldots x_n \in U_n - \overset{n-1}{\underset{i=1}{\cup}} U_i$. Now $\{x_n\}$ is a sequence in $X$. Let $x \in X$ be arbitrary, then $x \in U_j$ for some $j$. But by our choice of $\{x_n\}, x_i \notin U_j \ \forall\, i > j$. Therefore there exists no subsequence of $\{x_n\}$ which can pg**- congregates to $x$, which is a contradiction to X is sequentially pg**-compact. Therefore X is pg**- countably compact.

## 8. pg**- locally compact

**Definition 8.1** A subset $A$ of a topological space$(X,\tau)$ is said to be *relatively pg**-compact* if $pg^{**}cl(A)$ is pg**-compact in $X$.

**Definition 8.2** A topological space$(X,\tau)$ is said to be *pg**- locally compact* if for every point $x$ of X there is some pg**-compact subset $C$ of X that contains a pg**-neighbourhood of $x$ .

**Example 8.3** Let X has discrete topology. Then for every $x \in X, \{x\}$ is a pg**-neighbourhood of $x$ and pg**-compact. Therefore every discrete space is pg**-locally compact.

**Remark 8.4** A pg**-compact space is pg**- locally compact, but the converse is not true as seen in the following example.

**Example 8.5** Let X be an infinite discrete topological space. In this space for every $x \in X, \{x\}$ is a pg**-neighbourhood of $x$. Also $\{x\}$ is pg**-compact. Therefore X is pg**-locally compact, but it is not pg**-compact.

**Theorem 8.6** If $f:(X,\tau) \to (Y,\sigma)$ is pg**-irresolute, pg**-open map from a pg**-locally compact space $X$ onto a topological space $Y$, then $Y$ is pg**- locally compact.

**Proof** Let $y \in Y$ be arbitrary. Since $f$ is surjective we can find $x \in X$ such that $f(x) = y$. But $X$ is pg**- locally compact, therefore there exists a pg**-compact subset $C$ of X such that $C \supseteq U$ where $U$ is a pg**-neighbourhood of $x$. Therefore $x \in U \subseteq C$, this implies $y = f(x) \in f(U) \subseteq f(C)$ since $f$ is a pg**-open map, $f(U)$ is a pg**-

neighbourhood of $f(x)$. Also $f(C)$ is pg**-compact subset of $Y$ since $f$ is pg**-irresolute. Thus $Y$ is pg**- locally compact.

## 9. pg**- paracompact

**Definition 9.1** A collection $\mathcal{A} = \{U_\alpha\}_{\alpha\in\Delta}$ of subsets of a topological space $X$ is said to be *pg**- locally finite* if each point $x \in X$ has a pg**- neighbourhood having nonempty intersection with atmost finitely many members of $\mathcal{A}$.

**Example 9.2** Consider $\mathbb{R}$ with usual topology, then the collection of intervals $\mathcal{A} = \{(n, n+2)/\ n \in \mathbb{Z}\}$ is pg**- locally finite.

**Theorem 9.3** Let $(\mathrm{X}, \tau)$ be a topological space and let $\mathcal{A} = \{U_\alpha\}_{\alpha\in\Delta}$ be a pg**- locally finite family of subsets of $X$. Then $\mathcal{C} = \{pg^{**}cl(U_\alpha)\}_{\alpha\in\Delta}$ is also pg**- locally finite.

**Proof** Choose a point $x \in X$ and a pg**- neighbourhood $G$ of $x$ such that $U_\alpha \cap G = \varphi$ for all except for finitely many $\alpha \in \Delta$, then $U_\alpha \subset X - G$ this implies $pg^{**}cl(U_\alpha) \subset X - G$. Consequently $pg^{**}cl(U_\alpha) \cap G = \varphi$ for all except for finitely many $\alpha \in \Delta$. Therefore $\mathcal{C}$ is pg**- locally finite.

**Theorem 9.4** In a topological space, if the collection $\mathcal{A} = \{U_\alpha\}_{\alpha\in\Delta}$ is pg**- locally finite then $\cup \{pg^{**}cl(U_\alpha)\}_{\alpha\in\Delta} = pg^{**}cl(\cup \{U_\alpha\}_{\alpha\in\Delta})$.

**Proof** For every $\alpha \in \Delta$, we have $U_\alpha \subset \cup \{U_\alpha\}_{\alpha\in\Delta}$, then $pg^{**}cl(U_\alpha) \subset pg^{**}cl(\cup \{U_\alpha\}_{\alpha\in\Delta})$, this implies $\cup \{pg^{**}cl(U_\alpha)\}_{\alpha\in\Delta} \subset pg^{**}cl(\cup \{U_\alpha\}_{\alpha\in\Delta})$. On the other hand, let $x \in pg^{**}cl(\cup \{U_\alpha\}_{\alpha\in\Delta})$. Since $\mathcal{A}$ is pg**- locally finite, there exists a pg**- neighbourhood $G_x$ of $x$ such that $G_x$ has nonempty intersection with atmost finitely many members of $\mathcal{A}$. Let these finite members of $\mathcal{A}$ be $U_{\alpha_1}, U_{\alpha_2}, U_{\alpha_3}, \ldots, U_{\alpha_n}$. Therefore $G_x \cap U_\alpha \neq \varphi$ for $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$. Let $H_x$ be any pg**- neighbourhood of $x$ and let $I_x = G_x \cap H_x$. Now $I_x \cap (U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_n}) \neq \varphi$, since $x \in pg^{**}cl(\cup \{U_\alpha\}_{\alpha\in\Delta})$. But $I_x \subset H_x$, and $H_x$ is an arbitrary pg**- neighbourhood of $x$ and $x \in pg^{**}cl\big(U_{\alpha_1} \cup U_{\alpha_2} \cup \ldots \cup U_{\alpha_n}\big) = pg^{**}cl\big(U_{\alpha_1}\big) \cup pg^{**}cl\ (U_{\alpha_2}) \cup \ldots \cup pg^{**}cl\big(U_{\alpha_n}\big)$. Then $x \in pg^{**}cl(U_{\alpha_i})$ for some $i$ where $1 \leq i \leq n$. Therefore $x \in \cup \{pg^{**}cl(U_\alpha)\}_{\alpha\in\Delta}$. Therefore $\cup \{pg^{**}cl(U_\alpha)\}_{\alpha\in\Delta} = pg^{**}cl(\cup \{U_\alpha\}_{\alpha\in\Delta})$. Hence the theorem.

**Definition 9.5** Let $\mathcal{A}$ be a pg**-open covering of a topological space $X$. A collection $\mathcal{P}$ is called a *pg**-refinement* of $\mathcal{A}$ if $\mathcal{P}$ is also pg**-open covering of $X$ and each member of $\mathcal{P}$ is contained in some member of $\mathcal{A}$.

**Definition 9.6** A topological space $X$ is said to be *pg**- paracompact* if every pg**-open covering of $X$ has a pg**- locally finite pg**-refinement which is also a pg**-open covering of $X$.

**Example 9.7** Every pg**- compact space is pg**- paracompact. Since every finite subcover $\mathcal{P}$ of the pg**-open cover $\mathcal{A}$ is a pg**- locally finite pg**-refinement of $\mathcal{A}$.

**Theorem 9.8** Every pg**- closed subset of a pg**- paracompact space is pg**-paracompact.

**Proof** Let $A$ be a pg**- closed subset of the pg**- paracompact space $X$. Given a pg**- open covering $\{U_\alpha\}_{\alpha\in\Delta}$ of $A$. Let us form an pg**- open covering of $X$ by adjoining to $\{U_\alpha\}_{\alpha\in\Delta}$ the single pg**- open set $X - A$, that is $\{ \{U_\alpha\}_{\alpha\in\Delta} \cup (X - A) \}$. Take a pg**- locally finite pg**-refinement of this pg**- open covering of $X$ and intersect it with $A$. This gives a pg**- locally finite pg**-refinement of pg**- open covering $\{U_\alpha\}_{\alpha\in\Delta}$ of $A$.

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