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# Commuting Well bounded Operators on Banach Spaces

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**Abstract**

Well-bounded operators on Banach spaces are introduced by D. R. Smart [1].They coinside with the operators having, in some weaker sense,a spectral decomposition that converges only conditionally. T.A.Gillespie[2] proved that the sum of two commuting well-bounded operators is not always well-bounded even on Hilbert space.In this paper we give a general procedure to construct well-bounded operators on Banach space and show that the sum and product of well-bounded operators on Banach space need not be well-bounded operators.

**Keywords:** Banach Space, Scalar-type Operators, Well-bounded operators.

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## INTRODUCTION:

**Definition1:**

A bounded linear operator T on a complex reflexive Banach space is said to be well-bounded if there is a compact interval J = [a, b] and a positive constant M such that

$\| p(T) \| \leq M\{ \|p\|_J + \underset{J}{\mathrm{var}}\, p \}$ ---(I)

for every complex polynomial p, where $\|p\|_J$ denotes sup $\{| p(t) |: t \in J\}$ and

$\underset{J}{\mathrm{var}}\, p = \sup\{ \sum_{j=1}^{n} | p(\lambda_j) - p(\lambda_{j+1}) | / \lambda_0 = a, \lambda_1, \ldots\ldots, \lambda_n = b$ is any partition of J}

**Theorem 2**:

Suppose B is reflexive Banach space such that each closed linear subspace can be complemented then for every bounded linear operator T and any real number $\mu$ there is a bounded projection $P_\mu$ such that $R(P_\mu) = N(T - \mu I)$ and $P_\mu$ commutes with every bounded operator commuting with T.

Suppose X is a Banach space and $\{P_n\}$ is a sequence of bounded linear transformation on X satisfying the following:

1) Each $P_n$ is a projection on X. i.e. $P_n^2 = P_n$ .
2) $P_n P_m = 0$ , $n \neq m$.
3) $\sum_{n=1}^{n} P_n$ converges strongly to I.
4) $\| \sum_{j=1}^{n} P_{2j-1} \| \rightarrow \infty$ as $n \rightarrow \infty$

**Lemma 3:**

Let $\{P_n\}$ is a sequence of bounded linear transformations on a Banach space X satisfying (1),(2) ,(3) & (4) then $\| \sum_{j=1}^{n} P_{2j} \| \rightarrow \infty$ as $n \rightarrow \infty$ .

Proof: Put, $S_n(x) = \sum_{j=1}^{n} P_{2j} \sum_{j=1}^{n} P_j(x)$ , $x \in X$ , $n \geq 1$

Then we have by (3),

$$\text{s-} \lim_{n\to\infty} \sum_{j=1}^{n} P_j = I$$

i.e. $S_n(x) \rightarrow I(x) = x$, for every $x \in X$.

Hence ,$S_n(x)$ is bounded for every $x \in X$.

Therefore, by uniform boundedness theorem ,$\{S_n\}$ is uniformly bounded.

i.e. $\| S_n\| \leq M$ , $\forall\ n\geq 1$

Put, $A_n = \sum_{j=1}^{n} P_{2j-1}$ and $B_n = \sum_{j=1}^{n} P_{2j}$

Then, $\| A_n\| = \| A_n + B_n - B_n \|$

$\leq \| S_{2n} - B_n\|$

$\leq M +\| B_n\|$

As $\| A_n\| = \| \sum_{j=1}^{n} P_{2j-1}\| \to \infty$ as $n \to \infty$ ,

$\| B_n\| = \| \sum_{j=1}^{n} P_{2j}\| \to \infty$ as $n \to \infty$ .

**Lemma 4**:

Let $\{\lambda_n\}$ be a sequence in **R** such that $\sum_{j=1}^{n} P_j < \infty$ .Then the series $\sum_{n=1}^{\infty} \lambda_n P_n$ converges strongly.

**Proof:** Put, $Sn = \sum_{j=1}^{n} P_j$, $n \geq 1$

$\sum_{j=1}^{n} \lambda_j P_j = \lambda_{n+1} Sn + \sum_{j=1}^{n} (\lambda_j - \lambda_{j+1}) S_j$, $n\geq 1$ .........................(A)

We show that $\{\lambda_n\}$ is convergent.

for n> m, we have

$|\lambda_n - \lambda_m|$

$\leq |\lambda_n - \lambda_{n-1} + \lambda_{n-1} - \lambda_{n-2} + \text{------} + \lambda_{m+1} - \lambda_m|$

$\leq \sum_{j=1}^{n} P_j$

Since, $\sum_{j=1}^{n} P_j < \infty$, $\sum_{j=m}^{n-1} |\lambda_{j+1} - \lambda_j| \to 0$ as $n, m \to \infty$.

Thus, $\{\lambda_n\}$ is a Cauchy sequence and hence $\{\lambda_n\}$ is convergent .

Since $\{\lambda_{n+1}\}$ converges and $Sn \to I$ strongly, the sequence $\{\lambda_{n+1} S_n(x)\}$ converges for every $x \in X$.

Since $\{S_n(x)\}$ converges every $x \in X$, By uniform bounded theorem,

$\{S_n(x)\}$ is bounded.

Let , M be a positive constant such that $\| S_n \| \leq M$ , $n \geq 1$

Now for n> m, consider

$\|\sum_{j=1}^{n}(\lambda j - \lambda j + 1)Q_j$ - $\sum_{j=1}^{m}(\lambda j - \lambda j + 1)Q_j$ $\|$

$= \|\sum_{j=m+1}^{n}(\lambda j - \lambda j + 1)Q_j$ $\|$

$\leq \sum_{j=m+1}^{n} \| (\lambda j - \lambda j + 1)Q_j \|$

$\leq \sum_{j=m+1}^{n} |(\lambda_j - \lambda_{j+1})|$ $\|S_j\|$

$\leq M \sum_{j=m+1}^{n} | (\lambda j - \lambda j + 1)|$

$= M \sum_{j=m+1}^{n} |(\lambda_{j+1} - \lambda_j)|$

Since $\sum_{j=m+1}^{n} |(\lambda_{j+1} - \lambda_j)| \rightarrow 0$ as n, m $\rightarrow \infty$.

We get that $\sum_{j=1}^{n}(\lambda j - \lambda j + 1)Q_j$ $i$ is cauchy sequence .Therefore it converges in norm.

Now by using (A) ,we get that $\sum_{n=1}^{\infty} \lambda_n P_n$ is converges strongly.

**Lemma 5:**

Let $\{\lambda_n\}$ be a monotonic bounded sequence in R .Then

$(\sum_{j=1}^{n} \lambda_j P_j) = \sum_{j=1}^{n} p(\lambda_j) P_j + a_0 (I - S_n)$

for every complex polynomial $p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \dots\dots\dots + a_k\lambda^k$, $a_k \neq 0$.

**Proof:** Let $p(\lambda) = a_0 + a_1\lambda + a_2\lambda^2 + \dots\dots\dots + a_k\lambda^k$ , $a_k \neq 0$.

$$p\left(\sum_{j=1}^{n} \lambda_j P_j \lambda_j P_j\right)$$

$$= a_0 I + a_1 \sum_{j=1}^{k} \lambda_j P_j + a_2\left(\sum_{j=1}^{k} \lambda_j P_j\right)^2 + \text{-----------} + a_n\left(\sum_{j=1}^{k} \lambda_j P_j\right)^k$$

$$= a_0 I + a_1 \sum_{j=1}^{k} \lambda_j P_j + a_2\left(\left(\sum_{j=1}^{k} \lambda_i\right)^2 P_j + \text{---------} + a_n\left(\sum_{j=1}^{k} \lambda_i\right)^k P_j\right.$$

$$= a_0 I + \sum_{j=1}^{n} \left(a_1\lambda_j + a_2\lambda_j^2 + \dots\dots + a_k\lambda_j^k\right) P_j = a_0 I + \sum_{j=1}^{n} \left[p(\lambda_j) - a_0\right] P_j$$

$$= a_0 I + \sum_{j=1}^{n} p(\lambda_j) P_j - \sum_{j=1}^{n} a_0 P_j$$

$$= \sum_{j=1}^{n} p(\lambda_j) P_j + a_0\left(I - \sum_{j=1}^{n} P_j\right)$$

Therefore,

$$p\left(\sum_{j=1}^{n} \lambda_j P_j\right) = \sum_{j=1}^{n} p(\lambda_j) P_j + a_0(I - S_n)$$

**Lemma 6**:

Let $\{\lambda_n\}$ be a monotonic bounded sequence in R.Then the series $\sum_{n=1}^{\infty} \lambda_n P_n$ defines a well-bounded operator on X.

**Proof:** Let $\{\lambda_n\}$ be a monotonically increasing then

$$\sum_{j=1}^{n} |\lambda_{j+1} - \lambda_j| = \sum_{j=1}^{n} (|\lambda_{j+1} - \lambda_j)$$

$$= \lambda_2 - \lambda_1 + \lambda_3 - \lambda_2 + \text{----------} + \lambda_{n+1} - \lambda_n$$

$$= \lambda_{n+1} - \lambda_1$$

Since $\{\lambda_n\}$ is a monotonic and bounded ,it is convergent. i.e. $\{\lambda_{n+1} - \lambda_1\}$ converges.

Hence $\sum_{j=1}^{\infty} |\lambda_{j+1} - \lambda_j| < \infty$

Similarly ,if $\{\lambda_n\}$ be a monotonically decreasing then

$$\sum_{j=1}^{n} |\lambda_{j+1} - \lambda_j| = \sum_{j=1}^{n} (|\lambda_j - \lambda_{j+1})$$

$= \lambda_1 - \lambda_2 + \lambda_2 - \lambda_3 + \text{----------} + \lambda_n - \lambda_{n-1}$

$= \lambda_1 - \lambda_{n+1}$

which is convergent by the same argument.

i.e. $\sum_{j=1}^{\infty} |\lambda_{j+1} - \lambda_j| < \infty$

Thus by lemma (4) $\sum_{n=1}^{\infty} \lambda_n P_n$ converges strongly.

Define $T = s - \lim_{n\to\infty} \sum_{j=1}^{n} \lambda_j P_j$

Since $\{\lambda_n\}$ is bounded, we can choose a compact interval J=[a,b] such that $\{\lambda_n / n=1,2,.......\} \subset J$

Further for any complex polynomial $p(\lambda) = \sum_{i=0}^{k} a_i \lambda^i$ ,we have by lemma 5,

$$p\left(\sum_{j=1}^{n} \lambda_j P_j\right) = \sum_{j=1}^{n} p(\lambda_j) P_j + a_0 (I - S_n)$$

Hence

$$p\left(\sum_{j=1}^{n} \lambda_j P_j\right) x$$

$$= \sum_{j=1}^{n} p(\lambda_j) P_j x + a_0 (I - S_n) x$$

$=p(\lambda_{n+1})$ Sn x + $\sum p(\lambda_j) - p(\lambda_{j+1})$ $S_j$ x + p(0) (I - $S_n$)x

$\|p\ (\sum_{j=1}^{n} \lambda_j P_j\ )x\|$

$\leq |p(\lambda_{n+1})|\ \|S_n x\| + \sum_{j=1}^{n} |\ p(\lambda_j) - p(\lambda_{j+1})\ |\ \|S_j x\| + |p(0)|\|(x - S_n)$..................(B)

$\leq |\ p(\lambda_{n+1})|.\ \|\ S_n\ \|\ \|x\| + \sum_{j=1}^{n} |\ p(\lambda_j) - p(\lambda_{j+1})\ |\ .\ \|S_j\|\ \|x\ \|$

$+|p(0)\ |\ \|\ (x - S_n\ x)\|$

$\leq\ \|p\|_J\ .M\ \ \|x\|\ +\ \sum_{j=1}^{n}\ p\left(\lambda j\right) - p(\lambda j + 1\left.\right)\ |\ .\ M\ \ \|x\|\ + |p(0)|\ \|\ (x - S_n\ x)\|$

Since T $= \sum_{j=1}^{n} \lambda_j P_j$ converges strongly to T and p is continuous,

$$p\ (T_n)x\ \to p\ (T)x\ ,\ x \in\ X.$$

Hence by taking $n \to \infty$ in (B) we get

$\|\ p\ (T)x\ \|\ \ \leq\ (\ \ \|p\|_J\ +\ \ \operatorname*{var}_J p\ \ )\ M.\ \|x\|$

This implies that

$$\|\ p\ (T)\ \| \leq M\{\ \ \|p\|_J\ \|\ \ +\ \operatorname*{var}_J p\ \|x\|\|\ \ \ \ \}$$

Thus, T is well-bounded.

**Theorem 7:**

There exists well-bounded operators S,T with ST = TS, but S+T is not well bounded.

Proof: Define the sequences $\{\lambda_n\}$&$\{\mu_n\}$ are as follows:

$\lambda_k = \frac{2n+1}{2n}$ & $\mu_{2n} = \frac{2n-1}{2n}$ , $\lambda_k = \frac{2n}{2n-1}$ & $\mu_{2n=1} = \frac{2n-1}{2n}$ .

We show that $\{\lambda_n\}$ is decreasing

i.e. $\lambda_{k+1} \leq \lambda_k$ for all $k \geq 1$.

If k is even , k = 2n, we have

$$\lambda_k = \frac{2(n+1)}{2(n+1)-1} = \frac{2n+2}{2n+1}$$

And

$$\lambda_k - \lambda_k = \frac{2n+1}{2n}\frac{2n+1}{2n} - \frac{2(n+1)}{2n+1}$$

$$= 1 - \frac{1}{2n}1 + \frac{1}{2n} - 1 - \frac{1}{2n+1}$$

$$= \frac{1}{2n} - \frac{1}{2n+1} > 0.$$

Similarly ,If k is odd , k = 2n-1, then we have

$$\lambda_k - \lambda_k = \frac{2n}{2n-1} - \frac{2n+1}{2n}$$

$$= \frac{4n^2 - (4n^2 - 1)}{(2n-1)(2n)}$$

$$= \frac{1}{(2n-1)(2n)} > 0..$$

Hence, $\{\lambda_k\}$ is decreasing for all $k \geq 1$.

Now we show that $\{\mu_k\}$ is increasing. i.e. $\mu_{k+1} \geq \mu_k$ for all $k \geq 1$.

If k is even , k = 2n, we have

$$\mu_{2n} - \mu_{2n+1} = \frac{2n-1}{2n} - \frac{2(n+1)-1}{2(n+1)}$$

$$= \ 1-\frac{1}{2n} - 1 + \frac{1}{2(n+1)}$$

$$= \frac{1}{2n+2} - \frac{1}{2n} < 0 \lambda_k$$

Similarly , If k is odd , k = 2n-1, then we have

$$\mu_{2n-1} - \mu_{2n} = \frac{2(n-1)-1}{2(n-1)} - \frac{2n-1}{2n}$$

$$= \frac{2n-2-1}{2n-2} - \frac{2n-1}{2n}$$

$$= 1 - \frac{1}{2n-2} - 1 + \frac{1}{2n}$$

$$= \frac{1}{2n} - \frac{1}{2n-2} < 0.$$

Hence, $\{\mu_k\}$ is increasing.

If $S = \sum_{n=1}^{\infty} \lambda_n P_n \sum_{j=1}^{n} \lambda_j P_j$ and $T = \sum_{n=1}^{\infty} \mu_n P_n$

Then by lemma(6) S,T are well bounded operators on Banach space X with ST=TS.

We show that S+T is not well bounded.

Suppose that S+T is well bounded.

Since, $(S+T)x = \sum_{n=1}^{\infty} (\lambda_n + \mu_n) P_n \ x$ , for $x \in X$,

$(S+T)P_2x = \sum_{n=1}^{\infty} (\lambda_n + \mu_n) P_n \ P_2x$

$= (\lambda_2 + \mu_2) P_2x$

But

$$\lambda_{2n} + \mu_{2n} = \frac{2n+1}{2n} + \frac{2n-1}{2n} = 2 , \qquad n \geq 1.$$

Therefore, 2 is an eigenvalue of S+T.

By Theorem(2) there is a bounded projection Q which commutes with every operator commuting with S+T and $R(Q) = N((S+T) - 2I)$

Thus, $QP_n = P_n Q$ for all $n \geq 1$.

Also

$$(S+T)P_{2n}x = (\lambda_{2n} + \mu_{2n}) P_{2n} x$$

$$= 2 P_{2n} x$$

$P_{2n} x \in N((S+T) - 2I) = R(Q)$, $QP_{2n} x = P_{2n} x$, $x \in X$.

We also have

$$(S+T)P_{2n-1} x = (\lambda_{2n-1} + \mu_{2n-1}) P_{2n-1} x$$

$$= \alpha_n P_{2n-1} x$$

Where,

$$\alpha_n = \lambda_{2n-1}$$

$$= 1 + \frac{1}{2n-1} + 1 - \frac{1}{2n}$$

$$= 2 + \frac{1}{2n-1} - \frac{1}{2n}$$

$$\alpha_n = 2 + \frac{1}{2n(2n-1)} > 2 .$$

But $QP_{2n-1} x \in R(Q) = N((S+T) - 2I)$

$((S+T) - 2I)QP_{2n-1} x = 0$

i.e. $(S+T)P_{2n-1} x = 2 QP_{2n-1} x$

Also, $(S+T)Q P_{2n-1} x = Q(S+T) QP_{2n-1} x$

$$= Q(\lambda_{2n-1} + \mu_{2n-1}) P_{2n-1} x$$

$$= (\lambda_{2n-1} + \mu_{2n-1})\, QP_{2n-1}\, x$$

$$= \alpha_n Q\, P_{2n-1}\, x$$

Thus, $2QP_{2n-1} = \alpha_n Q\, P_{2n-1}\, x$ ,

where $\alpha_n > 2$ .Thus, $QP_{2n-1} = 0$

Thus ,$QP_{2n}x = P_{2n}\, x$ & $QP_{2n-1}\, x = 0$ , $x \in X$

i.e. $QP_{2n} = P_{2n}$ , $QP_{2n-1} = 0$ ,

Now $I = s\text{-} \lim_{n\to\infty} \sum_{k=1}^{n} P_k$

$$Q = s\text{-} \lim_{n\to\infty} \sum_{k=1}^{n} QP_k$$

$$= s\text{-} \lim_{n\to\infty} \sum_{k=1}^{n} QP_{2k}$$

$$= s\text{-} \lim_{n\to\infty} \sum_{k=1}^{n} P_{2k}$$

Thus, the partial sums of the series $\sum_{n=1}^{n} P_{2n}$ converges pointwise to Qx.

This implies that the sequence $\sum_{j=1}^{n} P_{2j}$ is pointwise bounded and by uniform boundedness, theorem the sequence is uniformly bounded. This leads to contradiction Since by lemma 1,

$\| \sum_{j=1}^{n} P_{2j} \| \to \infty$ as $n \to \infty$ .

This proves that S+T is not well-bounded.

**Theorem 8:**

There exists well-bounded operators S,T with ST=TS, but ST is not well bounded.

**Proof:**

Let S,T are well bounded operators defined as in theorem 7.

Then,S = $\sum_{n=1}^{\infty}\lambda_n P_n \sum_{j=1}^{n} \lambda_j P_j$ and T = $\sum_{n=1}^{\infty}\mu_n P_n$

and we have

$$(ST)(x) = S(T(x))$$

$$= S(\sum_{n=1}^{\infty}\mu_n P_n x)$$

$$= \sum_{n=1}^{\infty}\mu_n \sum_{m=1}^{\infty}\lambda_m P_m P_n x$$

$$= \sum_{n=1}^{\infty}\mu_n \lambda_n P_n x$$

Furthermore, for, $n \geq 1$

$$\mu_{2n-1} \cdot \lambda_{2n-1} = \frac{2n}{2n-1} \cdot \frac{2n-1}{2n} = 1$$

Hence.for any $x \in X$,

$$(ST)P_{2n-1}x = \mu_{2n-1} \cdot \lambda_{2n-1} P_{2n-1} x$$

$$= P_{2n-1}x$$

Since $\mu_{2n-1} \cdot \lambda_{2n-1} = 1$

Thus,1 is an eigenvalue of ST,Hence by theorem(A) there is a bounded projection U such that

R(U) = N(ST-I) and U commutes with every bounded operator commuting with ST.

$(ST)P_{2n-1}x = 0$, $x \in X$ ,

$P_{2n-1}x \in N((ST)-I) = R(U)$,

Thus

$U.P_{2n-1}x = P_{2n-1}x$

Also for any $x \in X$, $U.P_{2n}x \in R(U) = N((ST)-I)$

Therefore,

$$((ST)-I)UP_{2n}x = 0$$

i.e. $(ST)UP_{2n}x = UP_{2n}x$

But, $(ST)UP_{2n}x = U(ST)P_{2n}x$

$$= U(\lambda_{2n} \cdot \mu_{2n})P_{2n}x$$

$$= (\lambda_{2n}\mu_{2n})UP_{2n}x$$

$$= \beta_n UP_{2n}x$$

where, $\beta_n = \lambda_{2n}\mu_{2n} = \dfrac{4n^2-1}{4n^2} < 1$

Hence, $U.P_{2n}x = \beta_n UP_{2n}x$, $x \in X$.

$\beta_n < 1$, implies that $U.P_{2n}x = 0$, for every $x \in X$.

Thus, $U.P_{2n-1}x = P_{2n-1}x$ and $U.P_{2n}x = 0$, for every $x \in X$

Now $I = s - \lim_{n\to\infty} \sum_{k=1}^{n} P_k$

Thus, $U = s - \lim_{n\to\infty} \sum_{k=1}^{n} U.P_k$

$$= \text{ s - } \lim_{n\to\infty} \sum_{k=1}^{n} U.P_{2k-1}$$

$$= \text{ s - } \lim_{n\to\infty} \sum_{k=1}^{n} P_{2k-1}$$

This again gives contradiction to (4) that,

$\| \sum_{j=1}^{n} P_{2j-1} \| \rightarrow \infty$ as $n \rightarrow \infty$ .

This proves that ST is not well-bounded.

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