International Journal of Mathematics Research.
ISSN 0976-5840 Volume 9, Number 2 (2017), pp. 121-124

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Proof for Goldbach's Strong Conjecture

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Abstract

In this paper, we will give a simple solution/proof for the Goldbach's strong conjecture (called the "strong" or "binary").

Keywords: Prime number formula, Goldbach's strong conjecture

1. INTRODUCTION

Goldbach's original conjecture (sometimes called the "ternary" Goldbach conjecture), written in a June 7, 1742 letter to Euler, states "at least it seems that every number that is greater than 2 is the sum of three primes" (Goldbach 1742; Dickson 2005). Note that here Goldbach considered the number 1 to be a prime, a convention that is no longer followed. As re-expressed by Euler, an equivalent form of this conjecture (called the "strong" or "binary" Goldbach conjecture) asserts that all positive even integers ≥ 4 can be expressed as the sum of two primes. Two primes (p, q) such that $p + q = 2n$ for n a positive integer are sometimes called a Goldbach partition [1].

In this paper, we will give a simple solution to the Goldbach's strong conjecture. We will first make a general formula for primes (>2), and then evaluate all possibilities.

2. METHODS

Lemma: Every prime number can be written in the form $4k \pm 1$; where $k \in I^+$

Proof:

A number can be expressed as $(4k)$ or $(4k + 1)$ or $(4k + 2)$ or $(4k + 3)(or\ 4k - 1)$.

Since, all prime numbers (>2) are odd, therefore, we can rule out $(4k)$ and $(4k+2)$ as a possible expression for primes.

Thus, the only possible representations for primes are $(4k+1)$ or $(4k-1)$; $k \in I^+$.

Let's say we have 4 consecutive numbers $(a), (a+1), (a+2), (a+3)$.

Out of these four, there will be 2 alternate even numbers, and out of them, at least one will be divisible by 4 (as out of every 4 consecutive numbers, one of them will definitely be divisible by 4, and it has to be even).

Let the even numbers be $(a+1)$ and $(a+3)$.

<u>CASE I :</u> $(a+1)$ is divisible by 4

If $(a+2)$ is a prime number, it can be written as $(4p+1)$; where $(a+1) = 4p$; $p \in I^+$.

AND

If (a) is a prime number, it can be written as $(4p-1)$; where $(a+1) = 4p$; $p \in I^+$.

<u>CASE II :</u> $(a+3)$ is divisible by 4

If $(a+2)$ is a prime number, it can be written as $(4q-1)$; where $(a+3) = 4q$; $q \in I^+$.

AND

If (a) is a prime number, it can be written as $(4(q-1)+1)$; where $(a+3) = 4q$ and $(a-1) = 4(q-1)$; $q \in I^+$.

[since $(a+3)$ is a multiple of 4 $\Rightarrow$ $\{(a+3)-4 = (a-1)\}$ is also a multiple of 4]

Similarly, the same reasoning can be applied when taking (a) and $(a+2)$ as even.

Thus, prime numbers (>2) are of the form $(4k \pm 1)$; where $k \in I^+$.

Remark: Thus, every prime number, p (>2), can be written as:

$$p = (4k+1) \; or \; (4k-1) = (4k \pm 1); \text{where } k \in I^+$$

Examples: 3 = (4 x 1) – 1,

5 = (4 x 1) + 1,

7 = (4 x 2) – 1,

17389 = (4 x 4347) + 1, and so on.

***Conjecture*:** Every even integer greater than 2 can be expressed as the sum of two primes. [2]

Proof:

Let there be 2 prime numbers (>2), namely, p_1 and p_2.

$$\therefore p_1 = 4n \pm 1; n \in I^+ \quad (1)$$

$$p_2 = 4m \pm 1; m \in I^+ \quad (2)$$

$$\Rightarrow p_1 + p_2 = (4n \pm 1) + (4m \pm 1)$$

Thus, we have only 4 possibilities:

I. $(4n + 1) + (4m + 1)$

II. $(4n + 1) + (4m - 1)$

III. $(4n - 1) + (4m + 1)$

IV. $(4n - 1) + (4m - 1)$

Here, possibilities I and IV (same sign of 1s), and II and III (opposite signs of 1s) are similar.

CASE I : Considering possibilities I and IV

Therefore,

$$p_1 + p_2 = (4n + 1) + (4m + 1) \; or \; p_1 + p_2 = (4n - 1) + (4m - 1)$$

$$\Rightarrow p_1 + p_2 = 4(m + n) \pm 2$$

$$\Rightarrow p_1 + p_2 = 2[2(m + n) \pm 1]$$

$$\Rightarrow p_1 + p_2 = 2f; \; where \; f = [2(m + n) \pm 1]$$

$Thus, f \; is \; a \; positive \; odd \; integer (\because it \; is \; of \; the \; form \; 2q \pm 1) \; except \; 1 \; \{\because \; minimum \; value \; of \; m \; and \; n \; = 1 [from \; (1) and \; (2)] \; \Rightarrow \; minimum \; value \; of \; [2(m + n) \pm 1] = 3\}$

$$\therefore \; f \in \{3,5,7. \; . \; .\}$$

CASE II : Considering possibilities II and III

Therefore,

$$p_1 + p_2 = (4n + 1) + (4m - 1) \; or \; p_1 + p_2 = (4n - 1) + (4m + 1)$$

$$\Rightarrow p_1 + p_2 = 4(m + n)$$

$$\Rightarrow p_1 + p_2 = 2[2(m + n)]$$

$$\Rightarrow p_1 + p_2 = 2f'; \; where \; f' = [2(m + n)]$$

$Thus, f'$ $is\ a\ positive\ even\ integer(\because\ it\ is\ of\ the\ form\ 2q)\ except\ 2\ \{\because\ minimum$ $value\ of\ m\ and\ n = 1\ \ [from\ (1) and\ (2)]\ \Rightarrow minimum\ value\ of\ [2(m+n)] = 4\}$

$$\therefore\ f' \in \{4,6,8. . .\}$$

Hence, on combining cases I and II, we obtain

$$p_1 + p_2 = 2F\ ; where\ F \in f \cup f', i.e.\ F \in I^+ - \{1,2\}$$

$$\therefore\ F \in \{3,4,5,6. . .\}$$

$Thus, F\ is\ a\ positive\ integer\ except\ 1\ and\ 2, and\ minimum\ value\ of\ 2F = 6.$

$This\ means\ that\ every\ even\ integer\ (2F \in \{6,8,10. . .\})\ will\ have\ atleast\ one$ $pair\ of\ primes (p_1\ and\ p_2)\ such\ that\ p_1 + p_2 = 2F$

Thus, every even integer (>4) can be expressed as the sum of two primes.

The only case we are left with is – the even integer 4. We know that 4 can be expressed as the sum of two 2s, i.e. 4=2+2, thus, covering all the even integers (since 2 is a prime number, therefore, 4 is the sum of two primes).

3. CONCLUSION

Therefore, every even integer (>4) can be expressed as the sum of two primes (of the form $4k \pm 1$; where $k \in I^+$), and the even integer 4 can be expressed as the sum of two 2s***. Thus, every even integer greater than 2 can be expressed as the sum of two primes, and hence, proving the Goldbach's Strong Conjecture.***

4. REFERENCES

[1] Goldbach, C. Letter to L. Euler, June 7, 1742.

[2] Hazewinkel, Michiel, ed. (2001), "Goldbach problem", Encyclopedia of Mathematics, Springer, ISBN 978-1-55608-010-4