International Journal of Mathematics Research.
ISSN 0976-5840 Volume 12, Number 1 (2020), pp. 1-4

https://dx.doi.org/10.37624/IJMR/12.1.2020.1-4

Directed Strongly Regular Graphs and their Codes: Determinant Factorization for a Tight p-rank

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Abstract

The elementary divisors of the adjacency matrix A of a directed strongly regular graph are analysed to establish a tight bound for the rank over a finite field using the determinant factorization.
Keywords and phrases: Determinant, adjacency matrix, p-rank, directed strongly regular graphs, eigenvalues.

1. INTRODUCTION

This paper seeks to address the main open problem given in [1]. The open problem was trying to establish a tight bound for the rank of an adjacency matrix of a directed strongly regular graph over a finite field, which is equivalent to finding the dimension of the codes. The statement was conjectured in [1] as follows:

Conjecture 1. [1] If p^j divides $\det(A) \neq 0$ exactly, then the rank of A over $\mathbb{F}_p$ is $v - j$.

The conjecture was a hasty one and so there exists some counterexamples to it. The conjecture is modified and the modified form is recorded as Theorem 1 and a proof for it is given.

In this paper, we give a short background in Section 2, discuss some counterexamples in Section 3, give our results in Section 4 and then conclude in Section 5.

2. BACKGROUND

Suppose G is a directed graph on v vertices. Let A be a matrix of size v by v with entries from the set $\{0, 1\}$. Then A is the adjacency matrix of a directed strongly regular graph, if, for parameters (v, k, t, λ, μ), $0 < t < k$, and satisfies

- $\star$ $AJ = JA = kJ$
- $\star$ $A^2 = tI + \lambda A + \mu(J - I - A)$

where I and J are the identity matrix and the matrix of ones, respectively, both of order v. By Duval's [2] denotion, the parameters (v, k, t, λ, μ) of G denote that G is a directed graph on v vertices, such that every vertex has in-degree and out-degree k, the number of paths of length two from a vertices x to y is t if $x = y$, and the number of directed paths of length two directed from a vertex x to another vertex y is λ if there is an arc from x to y, and μ if there is no arc from x to y.

3. COUNTEREXAMPLE

The following counterexamples have their adjacency matrices given in [1]. Consider the adjacency matrix given by a tuple

- $(v, k, t, \lambda, \mu) = (24, 15, 11, 10, 8)$, which has eigenvalues $15, 3, -1$ with multiplicities $1, 2, 21$, respectively, elementary divisors

  $$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 45]$$

  and determinant $-135 = -3^3 \times 5$. We see that 3^j divides $-3^3 \times 5$ for $0 \leq j \leq 3$;

- $(v, k, t, \lambda, \mu) = (24, 9, 7, 2, 4)$, which has eigenvalues $9, 1, -3$ with multiplicities $1, 15, 8$, respectively, elementary divisors

  $$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 3, 3, 3, 3, 3, 3, 3, 27]$$

  and determinant $59049 = 3^{10}$. We see that 3^j divides 3^{10} for $0 \leq j \leq 10$;

- $(v, k, t, \lambda, \mu) = (18, 8, 5, 4, 3)$, which has eigenvalues $8, 2, -1$ with multiplicities $1, 3, 14$, respectively, elementary divisors

  $$[1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 1, 2, 2, 16]$$

  and determinant $64 = 2^6$. We see that 2^j divides 2^6 for $0 \leq j \leq 6$.

We have many possible choices for p^j such that p^j divides $\det(A) \neq 0$ which shouts out clearly that Conjecture 1 is false. The above points are some counterexamples to Conjecture 1. In Section 4, a modified statement of Conjecture 1 is given as Theorem 1.

4. THE RANK OF A USING DETERMINANT FACTORIZATION

In this section, we modify Conjecture 1 and prove it in some lemmas. The main result is Theorem 1.

Theorem 1. For k, r, s eigenvalues of the adjacency matrix, A, of a directed strongly regular graph, if krs is square free and there exists p^j such that $p^j \mid \det(A) \neq 0$ exactly, then the rank of A over $\mathbb{F}_p$ is $v - j$, where v is the order of A.

Lemma 1. The smallest invariant factor in the Smith normal form of A is 1. Equivalently, one of the eigenvalues k, r or s is ± 1.

Proof. The adjacency matrix has only zeros and ones in its first row, just as any other row. By the algorithm for the Smith normal form, we can have a 1 in the position $A_{(1,1)}$ and 0 in each of the positions $A_{(1,i)}$, $2 \leq i \leq v$. The result follows that the resulting Smith normal form of A will have a 1 among its invariant factors. □

Lemma 2. The determinant of A is plus or minus a product of powers of exactly two distinct primes.

Proof. By Lemma 1, we have that one of k, r or s is ± 1 and that all three eigenvalues are nonzero. Since $\det(A)$ is the product of powers of the eigenvalues, it follows that $\det(A)$ is plus or minus the product of the powers of exactly two distinct primes. □

Lemma 3. Let $\det(A) = \pm q_1^{\delta} q_2^{\psi}$ with q_1, q_2 distinct primes and δ and ψ positive integers. For p prime, if $p^j \mid \det(A)$ exactly, then $j = \delta$ or $j = \psi$.

Proof. If $p^j \mid \det(A)$ exactly, then $p^j = q_1^{\delta}$ or $p^j = q_2^{\psi}$. Since p, q_1 and q_2 are all prime numbers, the result follows. □

Now we give a proof of Theorem 1 below.

Proof. By Lemmas 1, 2 and 3, if $\det(A) \neq 0$, then $\det(A) = \pm q_1^{\delta} q_2^{\psi}$ with q_1, q_2 distinct primes and δ and ψ positive integers. The conditions that one of k, r or s is ± 1, by

Lemma 1, and that krs is square free implies q_1 and q_2 being distinct since krs is the greatest invariant factor. If $p^j \mid \det(A) \neq 0$ exactly, then

$$\frac{\det(A) \neq 0}{p^j} = \begin{cases} \pm q_2^{\psi} & \text{if } p^j = q_1^{\delta}, \\ \pm q_1^{\delta} & \text{if } p^j = q_2^{\psi}. \end{cases}$$

By definition, p-rank of A is the number of the invariant factors that are not divisible by p in the Smith normal form of A. Since $\det(A) \neq 0$, then rank $A = v$, and the p-rank of A is $v - \delta$ if $p^j = q_1^{\delta}$ or $v - \psi$ if $p^j = q_2^{\psi}$. By Lemma 3, $p^j = q_1^{\delta} \iff j = \delta$ and $p^j = q_2^{\psi} \iff j = \psi$. Therefore, the p-rank of A is $v - j$. $\square$

5. CONCLUSION

The rank over a finite field of the adjacency matrix A of a directed strongly regular graph was studied using the determinant bounds. It is easy to compute the spectrum of A using the formulas given in [3]. We provide counterexamples to a hasty conjecture, modify the conjecture and prove the modified one. It is not clear what the rank over a finite field of the adjacency matrix A would be if the largest invariant factor in the Smith normal form of A, krs, is not square-free. It will be interesting to establish a result for when krs is not square-free and in general, construct a link between the p-rank and the decoding properties of the codes.

Acknowledgement: The author acknowledge and appreciate a suggestion on the main theorem by Pal Hegedus.

REFERENCES

[1] A. Alahmadi, A. Alkenani, J.-L. Kim, M. Shi, and P. Solé, “Directed strongly regular graphs and their codes,” *Bull. Korean Math. Soc*, vol. 54, no. 2, pp. 497–505, 2017.

[2] A. M. Duval, “A directed graph version of strongly regular graphs,” *Journal of Combinatorial Theory, Series A*, vol. 47, no. 1, pp. 71–100, 1988.

[3] C. D. Godsil, S. A. Hobart, and W. J. Martin, “Representations of directed strongly regular graphs,” *European Journal of Combinatorics*, vol. 28, no. 7, pp. 1980–1993, 2007.