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A Subclass of Univalent Functions Defined by Ruscheweyh Operator

Timilehin G. Shaba

Department of Mathematics, University of Ilorin, Ilorin, Nigeria.

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Abstract

The main object of this paper is to introduce a new class $\mathcal{GJ}_n(m, \lambda, \alpha)$ define by Ruscheweyh operator involving function $f(z) \in \mathcal{A}_n$. Parallel results, for some related classes including the class of starlike, convex and Bazilevic functions respectively, are also obtained.

Keywords: Univalent function, starlike function, convex function, Bazilevic function, Ruscheweyh Operator.

1. INTRODUCTION AND DEFINITIONS

Let $\mathcal{A}_n$ denote the class of function of the form

$$f(z) = z + \sum_{k=n+1}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ and the space of holomorphic functions in $\mathbb{U}$, $n \in \mathbb{N} = \{1, 2, 3, \cdots\}$.
Let $\mathcal{J}_n$ denote the subclass of functions that are univalent in $\mathbb{U}$.
By $S_n^*(\alpha) \subset \mathcal{J}_n$ denote the subclass of starlike functions of order α, $0 \leq \alpha < 1$ which satisfies the condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}). \tag{1.2}$$

Further, a function $f(z)$ belonging to $\mathcal{K}_n(\alpha) \subset \mathcal{J}_n$ is said to be convex functions of order α, $0 \leq \alpha < 1$ in $\mathbb{U}$, if and only if

$$\Re\left(\frac{zf''(z)}{f'(z)} + 1\right) > \alpha \qquad (z \in \mathbb{U}), \tag{1.3}$$

and denote by $\mathcal{R}_n(\alpha)$ the class of functions in $\mathcal{J}_n$ which satisfy the condition

$$\Re\{f'(z)\} > \alpha \quad (z \in \mathbb{U}).$$

It is well known that $\mathcal{K}_n(\alpha) \subset S_n^*(\alpha) \subset \mathcal{J}_n$.

In [5] Ruscheweyh has define the operator

$$D^h : \mathcal{A}_n \longrightarrow \mathcal{A}_n,\ n \in \mathbb{N} = \{1, 2, 3, 4, \cdots\},$$
$$D^0 f(z) = f(z)$$
$$D^1 f(z) = zf'(z)$$
$$(h+1)D^{h+1}f(z) = z[D^h f(z)]' + hD^h f(z) \quad (z \in \mathbb{U}),$$

We note that if $f \in \mathcal{A}_n$, then

$$D^h f(z) = z + \sum_{k=n+1}^{\infty} \frac{(h+k-1)!}{h!(k-1)!} a_k z^k \quad (z \in \mathbb{U}). \tag{1.4}$$

where $h \in \mathbb{N} \cup \{0\} = \{0, 1, 2, 3, \ldots\}$.

To prove our main theorem we shall need the following lemma.

Lemma 1.1. [3] *Let q be analytic in $\mathbb{U}$ with $q(0) = 1$ and suppose that*

$$\Re\left(1 + \frac{zq'(z)}{q(z)}\right) > \frac{3\alpha - 1}{2\alpha} \quad (z \in \mathbb{U}).$$

Then $\Re(q(z)) > \alpha$ in $\mathbb{U}$ and $\frac{1}{2} \leq \alpha < 1$.

2. MAIN RESULTS

Definition 2.1. *A function $f(z) \in \mathcal{A}_n$ is said to be a member of the class $\mathcal{G}\mathcal{J}_n(h, \lambda, \alpha)$ if*

$$\left|\frac{D^{h+1}f(z)}{z}\left(\frac{D^h f(z)}{z}\right)^{\lambda} - 1\right| < 1 - \alpha, \quad z \in U \quad , \quad \lambda \geq -2 \quad and \quad \frac{1}{2} \leq \alpha < 1 \tag{2.1}$$

where D^h is the Ruscheweyh operator. Note that inequality (2.1) *implies that*

$$Re\left(\frac{D^{h+1}f(z)}{z}\left(\frac{D^h f(z)}{z}\right)^{\lambda}\right) > \alpha. \qquad \frac{1}{2} \leq \alpha < 1.$$

Remark 2.2. *The function family $\mathcal{GJ}_n(h,\lambda,\alpha)$ is a comprehensive class of analytic functions which includes some known and new classes of analytic and univalent functions. For example,*

1. $\mathcal{GJ}_n(0,-1,\alpha) \equiv S_n^*(\alpha)$,
2. $\mathcal{GJ}_n(1,-1,\alpha) \equiv \mathcal{K}_n(\alpha)$,
3. $\mathcal{GJ}_n(0,0,\alpha) \equiv \mathcal{R}_n(\alpha)$,
4. $\mathcal{GJ}_1(0,\lambda,\alpha) \equiv \mathcal{B}(\lambda,\alpha) =$
   $\left\{ f \in \mathcal{A} : \left| f'(z)\left(\frac{f(z)}{z}\right)^{\lambda} - 1 \right| < 1-\alpha; \lambda \geq -1, 0 \leq \alpha < 1,\ z \in \mathbb{U} \right\}$
   intoduced by Singh[4] *and studied by Babalola*[1].
5. $\mathcal{GS}_1(0,-2,\alpha) \equiv \mathcal{B}(\alpha)$ *studied by Frasin and Darus*[2].

Theorem 2.3. *If $f(z) \in \mathcal{A}_n$ satisfies the condition*

$$\Re\left(\frac{(h+2)D^{h+2}f(z)}{D^{h+1}f(z)} + \frac{\lambda(h+1)D^{h+1}f(z)}{D^h f(z)} - (1+\lambda)(h+1)\right) > \frac{3\alpha-1}{2\alpha} \tag{2.2}$$

then $f(z) \in \mathcal{GJ}_n(h,\lambda,\alpha)$

Proof. For $z \in \mathbb{U}$, define an analytic function $q(z)$ with $q(0) = 1$ by

$$q(z) = \frac{D^{n+1}f(z)}{z}\left(\frac{D^n f(z)}{z}\right)^{\lambda}$$

By simplification,

$$\ln q(z) = \ln(D^{h+1}f(z)) - \ln(z) + \lambda\ln(D^h f(z)) - \lambda\ln(z)$$

and by simple differentiation it implies that

$$\frac{q'(z)}{q(z)} = \frac{(D^{h+1}f(z))'}{(D^{h+1}f(z))} - \frac{1}{z} + \lambda\frac{(D^h f(z))'}{(D^h f(z))} - \lambda\frac{1}{z}$$

so that,

$$\Re\left(1 + \frac{zq'(z)}{q(z)}\right) = \Re\left(\frac{(h+2)D^{h+2}f(z)}{D^{h+1}f(z)} + \frac{\lambda(h+1)D^{h+1}f(z)}{D^h f(z)} - (1+\lambda)(h+1)\right) > \frac{3\alpha-1}{2\alpha}$$

which, by Lemma 1.1, implies

$$Re\left(\frac{D^{n+1}f(z)}{z}\left(\frac{D^n f(z)}{z}\right)^{\lambda}\right) > \alpha, \qquad \left(\frac{1}{2} \leq \alpha < 1\right).$$

□

Remark 2.4. *When $n = 1$ and $h = 0$ in Theorem 2.3, we have the following theorem.*

Theorem 2.5. [1] *If $f \in \mathcal{A}$ satisfies*

$$\Re\left\{\left(1+\frac{zf''(z)}{f'(z)}\right)+\lambda\frac{zf'(z)}{f(z)}\right\} > \lambda + \frac{3\alpha-1}{2\alpha}, \quad z \in \mathbb{U}$$

Then $\Re(q(z)) > \alpha$ in $\mathbb{U}$ and $\frac{1}{2} \leq \alpha < 1$.

From Theorem 2.3, the following Corollaries hold true.

Corollary 2.6. *If $f(z) \in \mathcal{A}_n$, and*

$$\Re\left\{\frac{4z^2f''(z)+z^3f'''(z)+2zf'(z)}{z^2f''(z)+2zf'(z)}\right\} > \frac{1}{2} \quad (z \in \mathbb{U}),$$

then

$$\Re\left[zf''(z)+f'(z)\right] > \frac{1}{2}, \quad (z \in \mathbb{U}).$$

Corollary 2.7. *If $f(z) \in \mathcal{A}_n$ and*

$$\Re\left\{\frac{zf''(z)}{f'(z)}-\frac{zf'(z)}{f(z)}\right\} > -\frac{3}{2} \quad (z \in \mathbb{U}),$$

then

$$\Re\left[\frac{zf'(z)}{f(z)}\right] > \frac{1}{2}.$$

That is $f(z)$ is starlike of order $\frac{1}{2}$, hence $f \in \mathcal{GJ}_n(0,-1,\frac{1}{2}) \equiv S_n^*(\alpha)$.

Corollary 2.8. *If $f(z) \in \mathcal{A}_n$ and*

$$\Re\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \frac{1}{2} \quad (z \in \mathbb{U}),$$

then

$$\Re\left[f'(z)\right] > \frac{1}{2}.$$

In another words, if the function $f(z)$ is convex of order $\frac{1}{2}$, then $f(z) \in \mathcal{GJ}_n(0,0,\frac{1}{2}) \equiv \mathcal{R}_n(\frac{1}{2})$

Corollary 2.9. *If $f(z) \in \mathcal{A}_n$ and*

$$\Re\left\{2\left(\frac{zf''(z)}{f'(z)}+1\right)-\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}),$$

then

$$\Re\left[\frac{z^{\frac{1}{2}}f'(z)}{f^{\frac{1}{2}}(z)}\right] > \frac{1}{2}.$$

That is $f(z)$ is Bazilevic of order $\frac{1}{2}$, type $\frac{1}{2}$ in $\mathbb{U}$.

Corollary 2.10. *If $f(z) \in \mathcal{A}_n$ and*

$$\Re\left\{2\left(\frac{zf''(z)}{f'(z)}+1\right)+\frac{zf'(z)}{f(z)}\right\}>1 \quad (z\in\mathbb{U}),$$

then

$$\Re\left[\frac{f^{\frac{1}{2}}(z)f'(z)}{z^{\frac{1}{2}}}\right]>\frac{1}{2}.$$

That is $f(z)$ is Bazilevic of order $\frac{1}{2}$, type $\frac{3}{2}$ in $\mathbb{U}$.

CONCLUSION

In this paper, using Ruscheweyh operator, we defined new subclass of univalent function and established some of its properties. Results obtained provide properties of certain subclasses of univalent functions.

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