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Variants of $\mathcal{R}$-Weakly Commuting and Reciprocal Continuous Mappings in $\mathfrak{H}$ -Metric space

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Abstract

In this paper, we prove common fixed point theorems for variants of R-weakly commuting and reciprocal mappings in $\mathfrak{H}$ -metric space that contains cubic and quadratic terms of distance function $\mathfrak{H}(x, y, z)$. At the end, we provide an example for the support.

Keywords and phrases: $\mathfrak{H}$ -metric space, $\mathcal{R}$-weakly commuting mapping, Reciprocal continuous mapping.

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1. INTRODUCTION

The Banach fixed point theorem is the fundamental method for studying fixed point theory, it states that every contraction mapping on a complete metric space has a unique fixed point. Let $(\mathcal{X}, d)$ be a complete metric space. If $\mathcal{T}: \mathcal{X} \to \mathcal{X}$ satisfies $d(\mathcal{T}(x), \mathcal{T}(y)) \leq k\, (d(x, y))$ for all $x, y \in \mathcal{X}$, $0 \leq k < 1$, then it has a unique fixed point. In 1969, Boyd and Wong [2] replaced the constant k in Banach contraction principle by a implicit function ψ and proved some fixed point theorems.

In 1997, Alber and Gueree-Delabriere [1] introduced the concept of weak contraction in metric space: A map $\mathcal{F}: \mathcal{X} \to \mathcal{X}$ is said to be weak contraction if for each $x, y \in \mathcal{X}$, there exists a function $\emptyset : [0, \infty) \to [0, \infty)$, $\emptyset(t) > 0$ for all $t > 0$ and $\emptyset(0) = 0$ such that $d(\mathcal{T}(x), \mathcal{T}(y)) \leq d(x, y) - \emptyset(d(x, y))$.

After that many author's have proved many common fixed point theorems using these type of contraction conditions in the literature.

In 1986 Jungck [7] introduced more generalized commutativity, so called compatibility. The notion of compatibility is an iterate of sequence.

Two self-mappings f and g on a metric space $(\mathcal{X}, d)$ are called compatible if $lim_n d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that

$$lim_n fx_n = lim_n gx_n = t, \text{for some } t \text{ in } \mathcal{X}.$$

In 1996, Jungck [4] introduced the notion of weakly compatible mappings and showed that compatible maps are weakly compatible, but converse may not be true.

Two self-mappings f and g on a metric space $(\mathcal{X}, d)$are called weakly compatible if they commute at their coincidence point i.e.,

if $fu = gu$ for some $u \in \mathcal{X}$ then $fgu = gfu$.

Two self-mappings f and g on a metric space $(\mathcal{X}, d)$ are called point wise $\mathcal{R}-$ weakly commuting on $\mathcal{X}$ if given $x \in \mathcal{X}$, there exists $\mathcal{R} > 0$ such that $d(fgx, gfx) \leq \mathcal{R}\, d(gx, fx)$ for all x in $\mathcal{X}$.

Remark 1.1 It is obvious that point wise $\mathcal{R}-$ weakly commuting maps commute at their coincidence points, but maps f and g can fail to be point wise $\mathfrak{R}$ -weakly commuting only if there exists some x in $\mathcal{X}$ such that $fx = gx$ but $fgx \neq gfx$. Therefore, the notion of point wise $\mathcal{R}$ -weak commutativity type mapping is equivalent to commutativity at coincidence points.

Definition 1.2 [18] Two self mappings f and g on a metric space $(\mathcal{X}, d)$ are said to be reciprocally continuous if $\lim_{n\to\infty} fgx_n = ft$ and $\lim_{n\to\infty} gfx_n = gt$,whenever $\{x_n\}$ is a sequence in $\mathcal{X}$ such that $fx_n = lim_n gx_n = t$ for some t in $\mathcal{X}$.

Remark 1.3 Continuous mappings are reciprocally continuous on $(\mathcal{X}, d)$, but the converse is not true.

2. PRELIMINARIES

In 2006, Zead Mustafa and Brailey Sims [10] introduced the notion of $\mathfrak{H}$-metric space as generalization of the concept of ordinary metric space.

Definition 2.1 [10] "A $\mathfrak{H}$-metric space is a pair $(\mathcal{X}, \mathfrak{H})$, where $\mathcal{X}$ is a non-empty set and $\mathfrak{H}$ is a non-negative real-valued function defined on $\mathcal{X} \times \mathcal{X} \times \mathcal{X}$ such that for all $x, y, z, a \in \mathcal{X}$, we have

(i) $\mathfrak{H}(x, y, z) = 0$ if $x = y = z$,

(ii) $0 < \mathfrak{H}(x, x, y)$, for all $x, y \in \mathcal{X}$, with $x \neq y$,

(iii) $\mathfrak{H}(x, x, y) \leq \mathfrak{H}(x, y, z)$,for all $x, y, z \in \mathcal{X}$, with $z \neq y$,

(iv) $\mathfrak{H}(x, y, z) = \mathfrak{H}(x, z, y) = \mathfrak{H}(y, z, x) = \cdots$, (symmetry in all three variables),

(v) $\mathfrak{H}(x, y, z) \leq \mathfrak{H}(x, a, a) + \mathfrak{H}(a, y, z)$, for all $x, y, z, a \in \mathcal{X}$ (rectangle inequality).

The function $\mathfrak{H}$ is called $\mathfrak{H}$-metric on $\mathcal{X}$."

Definition 2.2 [10] "A sequence $\{x_n\}$ in a $\mathfrak{H}$-metric space $\mathcal{X}$ is said to be convergent if there exist $x \in \mathcal{X}$ such that $\lim_{n,m\to\infty} \mathfrak{H}(x, x_n, x_m) = 0$ and one says that the sequence $\{x_n\}$ is $\mathfrak{H}$-convergent to x. We call x the limit of the sequence $\{x_n\}$ and write $x_n \to x$ or $\lim_{n\to\infty} x_n = x$."

Definition 2.3 [10] "In a $\mathfrak{H}$-metric space $\mathcal{X}$, a sequence $\{x_n\}$ is said to be $\mathfrak{H}$-Cauchy if given $\in > 0$, there is $n_0 \in \mathbb{N}$ such that $\mathfrak{H}(x_n, x_m, x_l) < \in$, for all $n, m, l \geq n_0$ i.e., $\mathfrak{H}(x_n, x_m, x_l) \to 0$ as $n, m, l \to \infty$."

Proposition 2.4 [10]" Let $\mathcal{X}$ be $\mathfrak{H}$-metric space. Then the following statements are equivalent:

(i) $\{x_n\}$ is $\mathfrak{H}$-convergent to x,

(ii) $\mathfrak{H}(x_n, x_n, x) \to 0$ as $n \to \infty$,

(iii) $\mathfrak{H}(x_n, x, x) \to 0$ as $n \to \infty$,

(iv) $\mathfrak{H}(x_m, x_n, x) \to 0$ as $n, m \to \infty$."

Proposition 2.5 [10]" Let $\mathcal{X}$ be $\mathfrak{H}$-metric space. Then the following statements are equivalent:

(i) The sequence $\{x_n\}$ is $\mathfrak{H}$-Cauchy;

(ii) For every $\in > 0$, there exists $n_0 \in \mathbb{N}$ such that $\mathfrak{H}(x_n, x_m, x_m) < \in$, $\forall n, m \geq n_0$."

3. POINTWISE $\mathcal{R}$ -WEAKLY COMMUTING AND RECIPROCAL CONTINUOUS MAPPINGS

In 2010, Manro [8] introduced the concept of weakly commuting, $\mathcal{R}$ -weakly commuting, $\mathcal{R}$ -weakly commuting maps of type(P) in $\mathfrak{H}$- metric space.

Definition 3.1 [8]" Two self-mappings f and g on a $\mathfrak{H}$- metric space $(\mathcal{X}, \mathfrak{H})$are called weakly commuting if $\mathfrak{H}(fgx, gfx, gfx) \leq \mathfrak{H}(fx, gx, gx)$, for all $x \in \mathcal{X}$."

Definition 3.2 [8]" Two self-mappings f and g on a $\mathfrak{H}$- metric space $(\mathcal{X}, \mathfrak{H})$ are called $\mathcal{R}$ -weakly commuting if there exist a positive real number $\mathcal{R}$ such that $\mathfrak{H}(fgx, fgx, gfx) \leq \mathcal{R}\, \mathfrak{H}(fx, fx, gx)$, for all $x \in \mathcal{X}$."

Remark 3.3 If $\mathcal{R} \leq 1$, then $\mathcal{R}$ -weakly commuting mappings are weakly commuting.

Definition 3.4 [22]" Two self-mappings f and g on a $\mathfrak{H}$- metric space $(\mathcal{X}, \mathfrak{H})$ are called compatible if, whenever $\{x_n\}$ in $\mathcal{X}$ such that $\{fx_n\}$ and $\{gx_n\}$ are $\mathfrak{H}$ -convergent to some $t \in \mathcal{X}$, then $\lim_{n \to \infty} \mathfrak{H}(fgx_n, fgx_n, gfx_n) = 0$."

Definition 3.5 [8]" Two self-mappings f and g on a $\mathfrak{H}$- metric space $(\mathcal{X}, \mathfrak{H})$ are called $\mathcal{R}$ -weakly commuting mappings of type(P) if there exist a positive real number $\mathcal{R}$ such that $\mathfrak{H}(ffx, ggx, ggx) \leq \mathcal{R}\, \mathfrak{H}(fx, gx, gx)$, for all $x \in \mathcal{X}$."

4. MAIN RESULTS

In 1994, Pant [17] defined the notion of $\mathcal{R}$-weakly commuting mappings in metric space to enlarge the scope of study of common fixed point theorems from class of compatible maps to the wider class of $\mathcal{R}$-weakly commuting mappings. These maps are not necessarily continuous at fixed point. In 2013, Murthy and Prasad [9] introduced a new type of inequality for a map that involves cubic terms of metric function $d(x, y)$ that extended and generalized the results of many cited in the literature of fixed point theory. In this section, we extend the result of Murthy and Prasad [9] for point wise $\mathcal{R}$-weakly commuting mappings and reciprocal continuous mapping satisfying a generalized weak contractive condition involving various combinations of $\mathfrak{H}$-metric functions

Theorem 4.1 Let $\mathcal{A}, \mathcal{B}, \mathcal{S}$ and $\mathcal{T}$are four self mappings of a complete $\mathfrak{H} -$ metric space $(\mathcal{X}, \mathfrak{H})$ satisfying the following conditions:

$(\boldsymbol{C_1})$ $\mathcal{S}(\mathcal{X}) \subset \mathcal{B}(\mathcal{X}), \mathcal{T}(\mathcal{X}) \subset \mathcal{A}(\mathcal{X})$;

$(\boldsymbol{C_2})$$(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$ are point wise $\mathcal{R}$-weakly commuting pairs;

$(\boldsymbol{C_3})$$(\mathcal{A}, \mathcal{S})$ and $(\mathcal{B}, \mathcal{T})$are compatible pairs of reciprocally continuous mappings;

$(\boldsymbol{C_4})$$[1 + \hbar \mathfrak{H}(\mathcal{A}p, \mathcal{B}q, \mathcal{B}q)]\mathfrak{H}^2(\mathcal{S}p, \mathcal{T}q, \mathcal{T}q) \leq$

$$\hbar\, max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\mathfrak{H}^2(\mathcal{A}p,\mathcal{S}p,\mathcal{S}p)\mathfrak{H}(\mathcal{B}q,\mathcal{T}q,\mathcal{T}q)\\+\mathfrak{H}(\mathcal{A}p,\mathcal{S}p,\mathcal{S}p)\mathfrak{H}^2(\mathcal{B}q,\mathcal{T}q,\mathcal{T}q)\end{array}\right],\\ \mathfrak{H}(\mathcal{A}p,\mathcal{S}p,\mathcal{S}p)\mathfrak{H}(\mathcal{A}p,\mathcal{T}q,\mathcal{T}q)\mathfrak{H}(\mathcal{B}q,\mathcal{S}p,\mathcal{S}p),\\ \mathfrak{H}(\mathcal{A}p,\mathcal{T}q,\mathcal{T}q)\mathfrak{H}(\mathcal{B}q,\mathcal{S}p,\mathcal{S}p)\mathfrak{H}(\mathcal{B}q,\mathcal{T}q,\mathcal{T}q)\end{array}\right\}$$

$$+\sigma(\mathcal{A}p,\mathcal{B}q)-\emptyset(\sigma(\mathcal{A}p,\mathcal{B}q),$$

where $\sigma(\mathcal{A}p,\mathcal{B}q)=max\left\{\begin{array}{c}\mathfrak{H}^2(\mathcal{A}p,\mathcal{B}q,\mathcal{B}q),\\ \mathfrak{H}(\mathcal{A}p,\mathcal{S}p,\mathcal{S}p)\mathfrak{H}(\mathcal{B}q,\mathcal{T}q,\mathcal{T}q),\\ \mathfrak{H}(\mathcal{A}p,\mathcal{T}q,\mathcal{T}q)\mathfrak{H}(\mathcal{B}q,\mathcal{S}p,\mathcal{S}p),\\ \frac{1}{2}\left[\begin{array}{c}\mathfrak{H}(\mathcal{A}p,\mathcal{S}p,\mathcal{S}p)\mathfrak{H}(\mathcal{A}p,\mathcal{T}q,\mathcal{T}q)+\\ \mathfrak{H}(\mathcal{B}q,\mathcal{S}p,\mathcal{S}p)\mathfrak{H}(\mathcal{B}q,\mathcal{T}q,\mathcal{T}q)\end{array}\right]\end{array}\right\}$

$\hbar \geq 0$ is a real number and $\emptyset$: $[0, \infty) \to [0, \infty)$ is a continuous function with $\emptyset(t) = 0$ iff $t = 0$ and $\emptyset(t) > 0$ for each $t > 0$.Then $\mathcal{A}u = \mathcal{B}u = \mathcal{S}u = \mathcal{T}u = u$ and u is a unique in $\mathcal{X}$.

Proof. Let $x_0 \in \mathcal{X}$. Using (C_1),we can find a point $x_1 \in \mathcal{X}$ such that $\mathcal{S}(x_0) = \mathcal{B}(x_1) = y_0$. For this point x_1, we can find another point $x_2 \in \mathcal{X}$ such that $y_1 = \mathcal{A}(x_2) = \mathcal{T}(x_1)$.

In general, one can construct a sequence $\{y_n\}$ in $\mathcal{X}$ such that

$$y_{2n} = \mathcal{S}(x_{2n}) = \mathcal{B}(x_{2n+1});$$

$$y_{2n+1} = \mathcal{T}(x_{2n+1}) = \mathcal{A}(x_{2n+2}), \quad \text{for each} \quad n \geq 0.$$

(4.2)

For brevity, we write $r_n = \mathfrak{H}(y_{n-1}, y_n, y_n)$.

Firstly, we will prove that r_n is non-increasing sequence and converges to 0.

Case I. If n is even, taking $p = x_{2n}$ and $q = x_{2n+1}$ in (C_4), we get

$$[1+\hbar\mathfrak{H}(\mathcal{A}x_{2n},\mathcal{B}x_{2n+1},\mathcal{B}x_{2n+1})]\mathfrak{H}^2(\mathcal{S}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}) \leq$$

$$\hbar\, max\left\{\begin{array}{c}\frac{1}{2}\left[\begin{array}{c}\mathfrak{H}^2(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\\+\mathfrak{H}(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}^2(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\end{array}\right],\\ \mathfrak{H}(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{A}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{S}x_{2n},\mathcal{S}x_{2n}),\\ \mathfrak{H}(\mathcal{A}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\end{array}\right\}$$

$$+\sigma(\mathcal{A}x_{2n},\mathcal{B}x_{2n+1})-\emptyset(\sigma(\mathcal{A}x_{2n},\mathcal{B}x_{2n+1}),$$

where $\sigma(\mathcal{A}x_{2n},\mathcal{B}x_{2n+1})=max\left\{\begin{array}{c}\mathfrak{H}^2(\mathcal{A}x_{2n},\mathcal{B}x_{2n+1},\mathcal{B}x_{2n+1}),\\ \mathfrak{H}(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}),\\ \mathfrak{H}(\mathcal{A}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{S}x_{2n},\mathcal{S}x_{2n}),\\ \frac{1}{2}\left[\begin{array}{c}\mathfrak{H}(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{A}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})+\\ \mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\end{array}\right]\end{array}\right\}$

Using (4.2), we get

$$[1+\hbar\mathfrak{H}(y_{2n-1},y_{2n},y_{2n})]\mathfrak{H}^2(y_{2n},y_{2n+1},y_{2n+1}) \le$$

$$\hbar\, max\left\{\begin{array}{c}\frac{1}{2}\begin{bmatrix}\mathfrak{H}^2(y_{2n-1},y_{2n},y_{2n})\mathfrak{H}(y_{2n},y_{2n+1},y_{2n+1})\\ +\mathfrak{H}(y_{2n-1},y_{2n},y_{2n})\mathfrak{H}^2(y_{2n},y_{2n+1},y_{2n+1})\end{bmatrix},\\ \mathfrak{H}(y_{2n-1},y_{2n},y_{2n})\mathfrak{H}(y_{2n-1},y_{2n+1},y_{2n+1})\mathfrak{H}(y_{2n},y_{2n},y_{2n}),\\ \mathfrak{H}(y_{2n-1},y_{2n+1},y_{2n+1})\mathfrak{H}(y_{2n},y_{2n},y_{2n})\mathfrak{H}(y_{2n},y_{2n+1},y_{2n+1})\end{array}\right\}$$

$$+\sigma(y_{2n-1},y_{2n})-\emptyset(\sigma(y_{2n-1},y_{2n}),$$

where $\sigma(y_{2n-1},y_{2n}) = max\left\{\begin{array}{c}\mathfrak{H}^2(y_{2n-1},y_{2n},y_{2n}),\\ \mathfrak{H}(y_{2n-1},y_{2n},y_{2n})\mathfrak{H}(y_{2n},y_{2n+1},y_{2n+1}),\\ \mathfrak{H}(y_{2n-1},y_{2n+1},y_{2n+1})\mathfrak{H}(y_{2n},y_{2n},y_{2n}),\\ \frac{1}{2}\begin{bmatrix}\mathfrak{H}(y_{2n-1},y_{2n},y_{2n})\mathfrak{H}(y_{2n-1},y_{2n+1},y_{2n+1})+\\ \mathfrak{H}(y_{2n},y_{2n},y_{2n})\mathfrak{H}(y_{2n},y_{2n+1},y_{2n+1})\end{bmatrix}\end{array}\right\}$

On putting $r_{2n} = \mathfrak{H}(y_{2n-1},y_{2n},y_{2n})$ we have

$$[1+\hbar r_{2n}]r_{2n+1}^2 \le \hbar max\left\{\frac{1}{2}[r_{2n}^2 r_{2n+1} + r_{2n}r_{2n+1}^2],0,0\right\}$$

$$+\sigma(y_{2n-1},y_{2n})-\emptyset(\sigma(y_{2n-1},y_{2n})),$$

where $\sigma(y_{2n-1},y_{2n}) = max\left\{r_{2n}^2, r_{2n}r_{2n+1}, 0, \frac{1}{2}[r_{2n}\mathfrak{H}(y_{2n-1},y_{2n+1},y_{2n+1})+0]\right\}$.

By using rectangular inequality and property of $\emptyset$, we get

$$\mathfrak{H}(y_{2n-1},y_{2n+1},y_{2n+1}) \le \mathfrak{H}(y_{2n-1},y_{2n},y_{2n})+\mathfrak{H}(y_{2n},y_{2n+1},y_{2n+1})$$

$= r_{2n}+r_{2n+1}$ and

$$\sigma(y_{2n-1},y_{2n}) \le r_1(x,y) = max\left\{r_{2n}^2, r_{2n}r_{2n+1}, 0, \frac{1}{2}[r_{2n}(r_{2n}+r_{2n+1}),0]\right\}.$$

If $r_{2n} < r_{2n+1}$, then we get

$\hbar r_{2n+1}^2 \le \hbar r_{2n+1}^2 - \emptyset(r_{2n+1}^2)$, a contradiction.

Therefore, $r_{2n+1}^2 \le r_{2n}^2$ i.e., $r_{2n+1} \le r_{2n}$.

Similarly, if *n* is odd, then we can obtain $r_{2n+2} < r_{2n+1}$.

It follows that the sequence $\{r_n\}$ is decreasing.

Let $\lim_{n\to\infty} r_n = x$, for some $x \ge 0$.

Suppose $x > 0$; then putting $p = x_{2n}$ and $q = x_{2n+1}$ in (C_4), we have

$$[1 + \hbar\mathfrak{H}(\mathcal{A}x_{2n}, \mathcal{B}x_{2n+1}, \mathcal{B}x_{2n+1})]\mathfrak{H}^2(\mathcal{S}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \leq$$

$$\hbar\, max\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \\ +\mathfrak{H}(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})\mathfrak{H}^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})\end{bmatrix}, \\ \mathfrak{H}(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})\mathfrak{H}(\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n}), \\ \mathfrak{H}(\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{matrix}\right\}$$

$$+ \sigma(\mathcal{A}x_{2n}, \mathcal{B}x_{2n+1}) - \emptyset(\sigma(\mathcal{A}x_{2n}, \mathcal{B}x_{2n+1}),$$

where $$\sigma(\mathcal{A}x_{2n}, \mathcal{B}x_{2n+1}) = max\left\{\begin{matrix} \mathfrak{H}^2(\mathcal{A}x_{2n}, \mathcal{B}x_{2n+1}, \mathcal{B}x_{2n+1}), \\ \mathfrak{H}(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}), \\ \mathfrak{H}(\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n}), \\ \frac{1}{2}\begin{bmatrix} \mathfrak{H}(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})\mathfrak{H}(\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) + \\ \mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})\end{bmatrix} \end{matrix}\right\}$$

Now by using triangular inequality and property of $\emptyset$ and proceeds limit $n \to \infty$, we get

$$[1 + \hbar x]x^2 \leq \hbar x^3 + x^2 - \emptyset(x^2).$$

This implies that $\emptyset(x^2) \leq 0$. Since x is positive, then by using the property of $\emptyset$, we get $x = 0$. Therefore, we conclude that

$$\lim_{n\to\infty} r_{2n} = \lim_{n\to\infty} \mathfrak{H}(y_{2n-1}, y_{2n}, y_{2n}) = x = 0. \quad (4.3)$$

Next, we show that $\{y_n\}$ is a Cauchy sequence. Suppose we assume that $\{y_n\}$ is not a Cauchy sequence. For a given $\in > 0$, we can find two sequences of positive integers $\{m(k)\}$ and $\{n(k)\}$ such that $n(k) > m(k) > k$,

$$\mathfrak{H}(y_{m(k)}, y_{n(k)}, y_{n(k)}) \geq \in, \ \mathfrak{H}(y_{m(k)}, y_{n(k)-1}, y_{n(k)-1}) < \in \quad (4.4)$$

Now $\in \leq \mathfrak{H}(y_{m(k)}, y_{n(k)}, y_{n(k)})$

$$\leq \mathfrak{H}(y_{m(k)}, y_{n(k)-1}, y_{n(k)-1}) + \mathfrak{H}(y_{n(k)-1}, y_{n(k)}, y_{n(k)})$$

Letting $k \to \infty$,we get $\lim_{k\to\infty} \mathfrak{H}(y_{m(k)}, y_{n(k)}, y_{n(k)}) = \in$ (4.5)

Now from the rectangular inequality, we have,

$$\left|\mathfrak{H}(y_{n(k)}, y_{m(k)+1}, y_{m(k)+1}) - \mathfrak{H}(y_{m(k)}, y_{n(k)}, y_{n(k)})\right|$$
$$\leq \mathfrak{H}(y_{m(k)}, y_{m(k)+1}, y_{m(k)+1})$$

Letting $k \to \infty$, and using (4.3) and (4.4), we get

$$\lim_{k\to\infty} \mathfrak{H}(y_{n(k)}, y_{m(k)+1}, y_{m(k)+1}) = \in \quad (4.6)$$

Now again from the rectangular inequality, we have,

$$\left|\mathfrak{H}(y_{m(k)+1}, y_{n(k)+1}, y_{n(k)+1}) - \mathfrak{H}(y_{m(k)}, y_{n(k)}, y_{n(k)})\right| \\ \leq \mathfrak{H}(y_{m(k)}, y_{m(k)+1}, y_{m(k)+1}) + \mathfrak{H}(y_{n(k)}, y_{n(k)+1}, y_{n(k)+1})$$

Letting $k \to \infty$, and using (4.3) and (4.4), we get

$$\lim_{k\to\infty} \mathfrak{H}(y_{n(k)+1}, y_{m(k)+1}, y_{m(k)+1}) = \in \qquad (4.7)$$

On putting $p = x_{m(k)}$ and $q = x_{n(k)}$ in (C_4), we get

$$[1 + h\mathfrak{H}(\mathcal{A}x_{m(k)}, \mathcal{B}x_{n(k)}, \mathcal{B}x_{n(k)})]\mathfrak{H}^2(\mathcal{S}x_{m(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}) \leq$$

$$h\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)})\mathfrak{H}(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}) \\ +\mathfrak{H}(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)})\mathfrak{H}^2(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}) \end{bmatrix}, \\ \mathfrak{H}(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)})\mathfrak{H}(\mathcal{A}x_{m(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)})\mathfrak{H}(\mathcal{B}x_{n(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)}), \\ \mathfrak{H}(\mathcal{A}x_{m(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)})\mathfrak{H}(\mathcal{B}x_{n(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)})\mathfrak{H}(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}) \end{Bmatrix}$$

$$+ \sigma(\mathcal{A}x_{m(k)}, \mathcal{B}x_{n(k)}) - \emptyset(\sigma(\mathcal{A}x_{m(k)}, \mathcal{B}x_{n(k)}),$$

where $\sigma(\mathcal{A}x_{m(k)}, \mathcal{B}x_{n(k)}) =$

$$max \begin{Bmatrix} \mathfrak{H}^2(\mathcal{A}x_{m(k)}, \mathcal{B}x_{n(k)}, \mathcal{B}x_{n(k)}), \\ \mathfrak{H}(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)})\mathfrak{H}(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}), \\ \mathfrak{H}(\mathcal{A}x_{m(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)})\mathfrak{H}(\mathcal{B}x_{n(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)}), \\ \frac{1}{2}\begin{bmatrix} \mathfrak{H}(\mathcal{A}x_{m(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)})\mathfrak{H}(\mathcal{A}x_{m(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}) + \\ \mathfrak{H}(\mathcal{B}x_{n(k)}, \mathcal{S}x_{m(k)}, \mathcal{S}x_{m(k)})\mathfrak{H}(\mathcal{B}x_{n(k)}, \mathcal{T}x_{n(k)}, \mathcal{T}x_{n(k)}) \end{bmatrix} \end{Bmatrix}$$

Using (4.2), we get

$$[1 + h\mathfrak{H}(y_{m(k)-1}, y_{n(k)-1}, y_{n(k)-1})]\mathfrak{H}^2(y_{m(k)}, y_{n(k)}, y_{n(k)}) \leq$$

$$h\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(y_{m(k)-1}, y_{m(k)}, y_{m(k)})\mathfrak{H}(y_{n(k)-1}, y_{n(k)}, y_{n(k)}) \\ +\mathfrak{H}(y_{m(k)-1}, y_{m(k)}, y_{m(k)})\mathfrak{H}^2(y_{n(k)-1}, y_{n(k)}, y_{n(k)}) \end{bmatrix}, \\ \mathfrak{H}(y_{m(k)-1}, y_{m(k)}, y_{m(k)})\mathfrak{H}(y_{m(k)-1}, y_{n(k)}, y_{n(k)})\mathfrak{H}(y_{n(k)-1}, y_{m(k)}, y_{m(k)}), \\ \mathfrak{H}(y_{m(k)-1}, y_{n(k)}, y_{n(k)})\mathfrak{H}(y_{n(k)-1}, y_{m(k)}, y_{m(k)})\mathfrak{H}(y_{n(k)-1}, y_{n(k)}, y_{n(k)}) \end{Bmatrix}$$

$$+\sigma(y_{m(k)-1}, y_{n(k)-1}) - \emptyset(\sigma(y_{m(k)-1}, y_{n(k)-1}),$$

where $\sigma(y_{m(k)-1}, y_{n(k)-1}) =$

$$max \begin{Bmatrix} \mathfrak{H}^2(y_{m(k)-1}, y_{n(k)-1}, y_{n(k)-1}), \\ \mathfrak{H}(y_{m(k)-1}, y_{m(k)}, y_{m(k)})\mathfrak{H}(y_{n(k)-1}, y_{n(k)}, y_{n(k)}), \\ \mathfrak{H}(y_{m(k)-1}, y_{n(k)}, y_{n(k)})\mathfrak{H}(y_{n(k)-1}, y_{m(k)}, y_{m(k)}), \\ \frac{1}{2}\begin{bmatrix} \mathfrak{H}(y_{m(k)-1}, y_{m(k)}, y_{m(k)})\mathfrak{H}(y_{m(k)-1}, y_{n(k)}, y_{n(k)}) + \\ \mathfrak{H}(y_{n(k)-1}, y_{m(k)}, y_{m(k)})\mathfrak{H}(y_{n(k)-1}, y_{n(k)}, y_{n(k)}) \end{bmatrix} \end{Bmatrix}$$

Letting $k \to \infty$, we get

$$[1 + \hbar \in] \in^2 \leq \hbar max \left\{ \frac{1}{2}[0 + 0], 0, 0 \right\} + \in^2 - \emptyset(\in^2)$$

$= \in^2 - \emptyset(\in^2)$, a contradiction.

Thus $\{y_n\}$ is a Cauchy sequence in $\mathcal{X}$.From the completeness of $\mathcal{X}$, there exists a $z \in \mathcal{X}$ such that $y_n \to z$ as $n \to \infty$.

Moreover, since

$y_{2n+1} = \mathcal{T}(x_{2n+1}) = \mathcal{A}(x_{2n+2})$ and $y_{2n} = \mathcal{S}(x_{2n}) = \mathcal{B}(x_{2n+1})$are subsequences of $\{y_n\}$, we obtain

$$\lim_{n\to\infty} \mathcal{T}(x_{2n+1}) = \lim_{n\to\infty} \mathcal{A}(x_{2n+2}) = \lim_{n\to\infty} \mathcal{S}(x_{2n}) = \lim_{n\to\infty} \mathcal{B}(x_{2n+1}) = z$$

If $\mathcal{B}$ and $\mathcal{T}$ are compatible, then

$$\lim_{n\to\infty} \mathfrak{H}(\mathcal{B}\mathcal{T}x_n, \mathcal{T}\mathcal{B}x_n, \mathcal{T}\mathcal{B}x_n) = 0\,;$$

that is, $\mathcal{B}z = \mathcal{T}z$. Also by the reciprocal continuity of $\mathcal{B}$ and T, we have

$\lim_{n\to\infty} \mathcal{B}\mathcal{T}x_{2n} = \mathcal{B}zand\lim_{n\to\infty} \mathcal{T}\mathcal{B}x_{2n} = \mathcal{T}z$.

Since $\mathcal{T}(\mathcal{X}) \subset \mathcal{A}(\mathcal{X})$, there exists a point w in $\mathcal{X}$ such that $\mathcal{T}z = \mathcal{A}w$.

Setting $p = w$ and $q = z$ in (C_4), we get

$$[1 + \hbar\mathfrak{H}(\mathcal{A}w, \mathcal{B}z, \mathcal{B}z)]\mathfrak{H}^2(\mathcal{S}w, \mathcal{T}z, \mathcal{T}z)$$
$$\leq \hbar\, max \left\{ \begin{matrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(\mathcal{A}w, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z) + \\ \mathfrak{H}(\mathcal{A}w, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}^2(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z) \end{bmatrix}, \\ \mathfrak{H}(\mathcal{A}w, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{A}w, \mathcal{T}z, \mathcal{T}z)\mathfrak{H}(\mathcal{B}z, \mathcal{S}w, \mathcal{S}w), \\ \mathfrak{H}(\mathcal{A}w, \mathcal{T}z, \mathcal{T}z)\mathfrak{H}(\mathcal{B}z, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z) \end{matrix} \right\}$$
$$+\sigma(\mathcal{A}w, \mathcal{B}z) - \emptyset(\sigma(\mathcal{A}w, \mathcal{B}z)),$$

$$\text{where} \quad \sigma(\mathcal{A}w, \mathcal{B}z) = max \left\{ \begin{matrix} \mathfrak{H}^2(\mathcal{A}w, \mathcal{B}z, \mathcal{B}z), \\ \mathfrak{H}(\mathcal{A}w, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z), \\ \mathfrak{H}(\mathcal{A}w, \mathcal{T}z, \mathcal{T}z)\mathfrak{H}(\mathcal{B}z, \mathcal{S}w, \mathcal{S}w), \\ \frac{1}{2}\begin{bmatrix} \mathfrak{H}(\mathcal{A}w, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{A}w, \mathcal{T}z, \mathcal{T}z) + \\ \mathfrak{H}(\mathcal{B}z, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z) \end{bmatrix} \end{matrix} \right\} = 0.$$

This implies that

$$[1 + \hbar\mathfrak{H}(\mathcal{T}z, \mathcal{T}z, \mathcal{T}z)]\mathfrak{H}^2(\mathcal{S}w, \mathcal{T}z, \mathcal{T}z)$$
$$\leq \hbar\, max \left\{ \begin{matrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(\mathcal{T}z, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{T}z, \mathcal{T}z, \mathcal{T}z) + \\ \mathfrak{H}(\mathcal{T}z, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}^2(\mathcal{T}z, \mathcal{T}z, \mathcal{T}z) \end{bmatrix}, \\ \mathfrak{H}(\mathcal{T}z, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{T}z, \mathcal{T}z, \mathcal{T}z)\mathfrak{H}(\mathcal{T}z, \mathcal{S}w, \mathcal{S}w), \\ \mathfrak{H}(\mathcal{T}z, \mathcal{T}z, \mathcal{T}z)\mathfrak{H}(\mathcal{T}z, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{T}z, \mathcal{T}z, \mathcal{T}z) \end{matrix} \right\} + 0 - \emptyset(0),$$

i.e., $\mathfrak{H}^2(\mathcal{S}w, \mathcal{T}z, \mathcal{T}z) \leq p\ max \begin{Bmatrix} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{Bmatrix} + 0 - \emptyset(0)$,

which implies that $\mathcal{S}w = \mathcal{T}z$, and hence $\mathcal{S}w = \mathcal{T}z = \mathcal{A}w = \mathcal{B}z$.

The point wise $\mathcal{R}$-weak commutativity of $\mathcal{B}$ and $\mathcal{T}$ implies that there exists an $\mathcal{R} > 0$ such that $\mathfrak{H}(\mathcal{BT}z, \mathcal{TB}z, \mathcal{TB}z) \leq \mathcal{R}\ \mathfrak{H}(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z)$,

which implies that $\mathcal{BT}z = \mathcal{TB}z$ and $\mathcal{TT}z = \mathcal{TB}z = \mathcal{BT}z = \mathcal{BB}z$.

Similarly, the point wise $\mathcal{R}$ -weak commutativity of $\mathcal{A}$ and $\mathcal{S}$ implies that there exists an $\mathcal{R} > 0$ such that $\mathfrak{H}(\mathcal{AS}w, \mathcal{SA}w, \mathcal{SA}w) \leq \mathcal{R}\ \mathfrak{H}(\mathcal{A}w, \mathcal{S}w, \mathcal{S}w)$, which implies that $\mathcal{AS}w = \mathcal{SA}w$ and $\mathcal{AA}w = \mathcal{AS}w = \mathcal{SA}w = \mathcal{SS}w$.

Again substituting $p = w$ and $q = \mathcal{T}z$ in (C_4), we get

$$[1 + \hbar\mathfrak{H}(\mathcal{A}w, \mathcal{BT}z, \mathcal{BT}z)]\mathfrak{H}^2(\mathcal{S}w, \mathcal{TT}z, \mathcal{TT}z)$$
$$\leq \hbar\ max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(\mathcal{A}w, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{BT}z, \mathcal{TT}z, \mathcal{TT}z) + \\ \mathfrak{H}(\mathcal{A}w, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}^2(\mathcal{BT}z, \mathcal{TT}z, \mathcal{TT}z) \end{bmatrix}, \\ \mathfrak{H}(\mathcal{A}w, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{A}w, \mathcal{TT}z, \mathcal{TT}z)\mathfrak{H}(\mathcal{BT}z, \mathcal{S}w, \mathcal{S}w), \\ \mathfrak{H}(\mathcal{A}w, \mathcal{TT}z, \mathcal{TT}z)\mathfrak{H}(\mathcal{BT}z, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{BT}z, \mathcal{TT}z, \mathcal{TT}z) \end{Bmatrix}$$
$$+\sigma(\mathcal{A}w, \mathcal{BT}z) - \emptyset(\sigma(\mathcal{A}w, \mathcal{BT}z)),$$

where $$\sigma(\mathcal{A}w, \mathcal{BT}z) = max \begin{Bmatrix} \mathfrak{H}^2(\mathcal{A}w, \mathcal{BT}z, \mathcal{BT}z), \\ \mathfrak{H}(\mathcal{A}w, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{BT}z, \mathcal{TT}z, \mathcal{TT}z), \\ \mathfrak{H}(\mathcal{A}w, \mathcal{TT}z, \mathcal{TT}z)\mathfrak{H}(\mathcal{BT}z, \mathcal{S}w, \mathcal{S}w), \\ \frac{1}{2}\begin{bmatrix} \mathfrak{H}(\mathcal{A}w, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{A}w, \mathcal{TT}z, \mathcal{TT}z) + \\ \mathfrak{H}(\mathcal{BT}z, \mathcal{S}w, \mathcal{S}w)\mathfrak{H}(\mathcal{BT}z, \mathcal{TT}z, \mathcal{TT}z) \end{bmatrix} \end{Bmatrix}.$$

On simplification we have

$$[1 + \hbar\mathfrak{H}(\mathcal{T}z, \mathcal{TT}z, \mathcal{TT}z)]\mathfrak{H}^2(\mathcal{T}z, \mathcal{TT}z, \mathcal{TT}z) \leq \hbar max \begin{Bmatrix} \frac{1}{2}[0+0], \\ 0, \\ 0 \end{Bmatrix}$$
$$+\mathfrak{H}^2(\mathcal{T}z, \mathcal{TT}z, \mathcal{TT}z) - \emptyset(\mathfrak{H}^2(\mathcal{T}z, \mathcal{TT}z, \mathcal{TT}z)).$$

Hence $\mathcal{T}z = \mathcal{TT}z$. Thus $\mathcal{T}z = \mathcal{TT}z = \mathcal{BT}z$.

Therefore, $\mathcal{T}z$ is a common fixed point of $\mathcal{B}$ and $\mathcal{T}$.

Taking $p = \mathcal{S}w$ and $q = z$ in (C_4), we get

$$[1 + \hbar\mathfrak{H}(\mathcal{A}\mathcal{S}w, \mathcal{B}z, \mathcal{B}z)]\mathfrak{H}^2(\mathcal{S}\mathcal{S}w, \mathcal{T}z, \mathcal{T}z)$$

$$\leq \hbar \, max \left\{ \begin{array}{c} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(\mathcal{A}\mathcal{S}w, \mathcal{S}\mathcal{S}w, \mathcal{S}\mathcal{S}w)\mathfrak{H}(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z) + \\ \mathfrak{H}(\mathcal{A}\mathcal{S}w, \mathcal{S}\mathcal{S}w, \mathcal{S}\mathcal{S}w)\mathfrak{H}^2(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z) \end{bmatrix}, \\ \mathfrak{H}(\mathcal{A}\mathcal{S}w, \mathcal{S}\mathcal{S}w, \mathcal{S}\mathcal{S}w)\mathfrak{H}(\mathcal{A}\mathcal{S}w, \mathcal{T}z, \mathcal{T}z)\mathfrak{H}(\mathcal{B}z, \mathcal{S}\mathcal{S}w, \mathcal{S}\mathcal{S}w), \\ \mathfrak{H}(\mathcal{A}\mathcal{S}w, \mathcal{T}z, \mathcal{T}z)\mathfrak{H}(\mathcal{B}z, \mathcal{S}\mathcal{S}w, \mathcal{S}\mathcal{S}w)\mathfrak{H}(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z) \end{array} \right\}$$

$$+\sigma(\mathcal{A}\mathcal{S}w, \mathcal{B}z) - \emptyset(\sigma(\mathcal{A}\mathcal{S}w, \mathcal{B}z)),$$

where $\sigma(\mathcal{A}\mathcal{S}w, \mathcal{B}z) = max \left\{ \begin{array}{c} \mathfrak{H}^2(\mathcal{A}\mathcal{S}w, \mathcal{B}z, \mathcal{B}z), \\ \mathfrak{H}(\mathcal{A}\mathcal{S}w, \mathcal{S}\mathcal{S}w, \mathcal{S}\mathcal{S}w)\mathfrak{H}(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z), \\ \mathfrak{H}(\mathcal{A}\mathcal{S}w, \mathcal{T}z, \mathcal{T}z)\mathfrak{H}(\mathcal{B}z, \mathcal{S}\mathcal{S}w, \mathcal{S}\mathcal{S}w), \\ \frac{1}{2}\begin{bmatrix} \mathfrak{H}(\mathcal{A}\mathcal{S}w, \mathcal{S}\mathcal{S}w, \mathcal{S}\mathcal{S}w)\mathfrak{H}(\mathcal{A}\mathcal{S}w, \mathcal{T}z, \mathcal{T}z) + \\ \mathfrak{H}(\mathcal{B}z, \mathcal{S}\mathcal{S}w, \mathcal{S}\mathcal{S}w)\mathfrak{H}(\mathcal{B}z, \mathcal{T}z, \mathcal{T}z) \end{bmatrix} \end{array} \right\}$

On solving, we have

$$[1 + \hbar\mathfrak{H}(\mathcal{S}\mathcal{S}w, \mathcal{S}w, \mathcal{S}w)]\mathfrak{H}^2(\mathcal{S}\mathcal{S}w, \mathcal{S}w, \mathcal{S}w)$$

$$\leq \hbar \, max \left\{ \begin{array}{c} \frac{1}{2}[0 + 0], \\ 0, \\ 0 \end{array} \right\} + \mathfrak{H}^2(\mathcal{S}w, \mathcal{S}\mathcal{S}w, \mathcal{S}\mathcal{S}w) - \emptyset(\mathfrak{H}^2(\mathcal{S}w, \mathcal{S}\mathcal{S}w, \mathcal{S}\mathcal{S}w)).$$

Hence $\mathcal{S}w = \mathcal{S}\mathcal{S}w$. Thus $\mathcal{S}w = \mathcal{S}\mathcal{S}w = \mathcal{A}\mathcal{A}w$,

Thus $\mathcal{S}w$ is a common fixed point of $\mathcal{A}$ and $\mathcal{S}$.

If $\mathcal{S}w = \mathcal{T}z = u$, then$\mathcal{T}u = \mathcal{B}u = \mathcal{S}u = \mathcal{A}u = u$. Hence u is a common fixed point of $\mathcal{A}, \mathcal{B}, \mathcal{S}$and $\mathcal{T}$.

Uniqueness: Suppose that $v \neq u$ are two common fixed points of $\mathcal{A}, \mathcal{B}, \mathcal{S}$and $\mathcal{T}$.

On putting$p = u$ and $q = v$ in (C_4), we have

$[1 + \hbar\mathfrak{H}(\mathcal{A}u, \mathcal{B}v, \mathcal{B}v)]\mathfrak{H}^2(\mathcal{S}u, \mathcal{T}v, \mathcal{T}v) \leq \hbar \, max\{0,0,0\} + \sigma(\mathcal{A}u, \mathcal{B}v) - \emptyset(\sigma(\mathcal{A}u, \mathcal{B}v))$.

i.e., $[1 + \hbar\mathfrak{H}(u, v, v)]\mathfrak{H}^2(u, v, v) \leq \hbar \, max\{0,0,0\} + \mathfrak{H}^2(u, v, v) - \emptyset(\mathfrak{H}^2(u, v, v))$

$i.e., \mathfrak{H}^2(u, v, v) = 0$, This implies $u = v$.

This completes the proof.

5. R- WEAKLY COMMUTING MAPPINGS OF TYPE (P).

In 2009, Kumar et al. [10] defined the concept of $\mathcal{R}$- weakly commuting mappings of type (P) in metric spaces and proved a common fixed point theorem using these mappings.

Now we prove a common fixed point theorem for pairs of $\mathcal{R}$- weakly commuting mappings of type (P) satisfying a weak contraction condition that involves various combinations of the metric functions in $\mathfrak{H} -$ metric space.

Theorem 5.1 Let $\mathcal{S}$,$\mathcal{T}$, $\mathcal{A}$ and $\mathcal{B}$are four self mappings of a complete $\mathfrak{H}-$ metric space $(\mathcal{X},\mathfrak{H})$

into itself satisfying (C_1), (C_4) and the following condition:

$(\mathcal{A},\mathcal{S})$ and $(\mathcal{B},\mathcal{T})$ are $\mathcal{R}$- weakly commuting of type (P),

Then $\mathcal{S},\mathcal{T},\mathcal{A}$ and $\mathcal{B}$ have a unique common fixed point.

Proof. Let $x_0 \in \mathcal{X}$. Using (C_1),we can find a point $x_1 \in \mathcal{X}$ such that $\mathcal{S}(x_0) = \mathcal{B}(x_1) = y_0$. For this point x_1, we can find another point $x_2 \in \mathcal{X}$ such that $y_1 = \mathcal{A}(x_2) = \mathcal{T}(x_1)$. In general, one can construct a sequence $\{y_n\}$ in $\mathcal{X}$such that

$y_{2n} = \mathcal{S}(x_{2n}) = \mathcal{B}(x_{2n+1})$;

$y_{2n+1} = \mathcal{T}(x_{2n+1}) = \mathcal{A}(x_{2n+2})$ for each $n \geq 0$.

From Theorem 4.1, $\{y_n\}$ is a Cauchy sequence in $\mathcal{X}$. From the completeness of $\mathcal{X}$, there exists a $z \in \mathcal{X}$ such that $y_n \to z$ as $n \to \infty$. Moreover, since

$y_{2n+1} = \mathcal{T}(x_{2n+1}) = \mathcal{A}(x_{2n+2})$ and $y_{2n} = \mathcal{S}(x_{2n}) = \mathcal{B}(x_{2n+1})$are subsequences of $\{y_n\}$, we obtain

$$\lim_{n\to\infty} \mathcal{T}(x_{2n+1}) = \lim_{n\to\infty} \mathcal{A}(x_{2n+2}) = \lim_{n\to\infty} \mathcal{S}(x_{2n}) = \lim_{n\to\infty} \mathcal{B}(x_{2n+1}) = z.$$

.

Case 1: Suppose that $\mathcal{A}$ is continuous. Then $\{\mathcal{A}\mathcal{A}x_{2n}\}$and $\{\mathcal{A}\mathcal{S}x_{2n}\}$ converges to $\mathcal{A}z$ as $n \to \infty$. Since the mappings $\mathcal{A}$ and $\mathcal{S}$ are $\mathcal{R}$-weakly commuting of type (P), we have

$\mathfrak{H}(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}, \mathcal{S}\mathcal{S}x_{2n}) \leq \mathcal{R}\, \mathfrak{H}(\mathcal{A}x_{2n}, \mathcal{S}x_{2n}, \mathcal{S}x_{2n})$.

Lettingn $n \to \infty$, we get $\lim_{n\to\infty} \mathcal{S}\mathcal{A}x_{2n} = \mathcal{A}z$.

On putting$p = \mathcal{A}x_{2n}$ and $q = x_{2n+1}$ in (C_4), we get

$$[1 + \hbar\mathfrak{H}(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{B}x_{2n+1}, \mathcal{B}x_{2n+1})]\mathfrak{H}^2(\mathcal{S}\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})$$

$$\leq \hbar\, max \left\{ \begin{array}{c} \frac{1}{2}\left[\begin{array}{l} \mathfrak{H}^2(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \\ +\mathfrak{H}(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n})\mathfrak{H}^2(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{array} \right], \\ \mathfrak{H}(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n})\mathfrak{H}(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n}), \\ \mathfrak{H}(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{S}\mathcal{A}x_{2n}, \mathcal{S}\mathcal{A}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1}, \mathcal{T}x_{2n+1}, \mathcal{T}x_{2n+1}) \end{array} \right\}$$

$$+\sigma(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{B}x_{2n+1}) - \emptyset(\sigma(\mathcal{A}\mathcal{A}x_{2n}, \mathcal{B}x_{2n+1})),$$

where

$$\sigma(\mathcal{A}\mathcal{A}x_{2n},\mathcal{B}x_{2n+1}) = max\left\{\begin{matrix} \mathfrak{H}^2(\mathcal{A}\mathcal{A}x_{2n},\mathcal{B}x_{2n+1},\mathcal{B}x_{2n+1}), \\ \mathfrak{H}(\mathcal{A}\mathcal{A}x_{2n},\mathcal{S}\mathcal{A}x_{2n},\mathcal{S}\mathcal{A}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}), \\ \mathfrak{H}(\mathcal{A}\mathcal{A}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{S}\mathcal{A}x_{2n},\mathcal{S}\mathcal{A}x_{2n}), \\ \frac{1}{2}\left[\begin{matrix}\mathfrak{H}(\mathcal{A}\mathcal{A}x_{2n},\mathcal{S}\mathcal{A}x_{2n},\mathcal{S}\mathcal{A}x_{2n})\mathfrak{H}(\mathcal{A}\mathcal{A}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}) \\ +\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{S}\mathcal{A}x_{2n},\mathcal{S}\mathcal{A}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\end{matrix}\right]\end{matrix}\right\}.$$

Letting limit as $n \to \infty$, we have

$$[1 + \hbar\mathfrak{H}(\mathcal{A}z,z,z)]\mathfrak{H}^2(\mathcal{A}z,z,z) \leq \hbar\, max\left\{\begin{matrix}\frac{1}{2}[0+0], \\ 0, \\ 0\end{matrix}\right\} + \mathfrak{H}^2(\mathcal{A}z,z,z) - \emptyset(\mathfrak{H}^2(\mathcal{A}z,z,z)),$$

i.e., $[1 + \hbar\mathfrak{H}(\mathcal{A}z,z,z)]\mathfrak{H}^2(\mathcal{A}z,z,z) \leq \mathfrak{H}^2(\mathcal{A}z,z,z) - \emptyset(\mathfrak{H}^2(\mathcal{A}z,z,z))$.

This implies $\mathfrak{H}^2(\mathcal{A}z,z,z) = 0$, i.e., $\mathcal{A}z = z$.

Next, we shall show that $\mathcal{S}z = z$.

For this, putting $p = z$ and $q = x_{2n+1}$ in (C_4) and taking limit as $n \to \infty$ we get,

$$[1 + \hbar\mathfrak{H}(\mathcal{A}z,\mathcal{B}x_{2n+1},\mathcal{B}x_{2n+1})]\mathfrak{H}^2(\mathcal{S}z,\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}) \leq \hbar\, max\left\{\begin{matrix}\frac{1}{2}\left[\begin{matrix}\mathfrak{H}^2(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(z,z,z) \\ +\mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}^2(z,z,z)\end{matrix}\right], \\ \mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{A}z,z,z)\mathfrak{H}(z,\mathcal{S}z,\mathcal{S}z), \\ \mathfrak{H}(\mathcal{A}z,z,z)\mathfrak{H}(z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(z,z,z)\end{matrix}\right\}$$

$$+\sigma(\mathcal{A}z,z) - \emptyset(\sigma(\mathcal{A}z,z)),$$

where $\sigma(\mathcal{A}z,z) = max\left\{\begin{matrix}\mathfrak{H}^2(\mathcal{A}z,z,z), \mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(z,z,z), \\ \mathfrak{H}(\mathcal{A}z,z,z)\mathfrak{H}(z,\mathcal{S}z,\mathcal{S}z), \\ \frac{1}{2}\left[\begin{matrix}\mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{A}z,z,z) \\ +\mathfrak{H}(z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(z,z,z)\end{matrix}\right]\end{matrix}\right\} = 0.$

Therefore,

$$[1 + \hbar\mathfrak{H}(z,z,z)]\mathfrak{H}^2(\mathcal{S}z,z,z) \leq \hbar\, max\left\{\begin{matrix}\frac{1}{2}[0+0], \\ 0, \\ 0\end{matrix}\right\} + 0 - \emptyset(0).$$

Thus, $\mathfrak{H}^2(\mathcal{S}z,z,z) = 0$, implies $\mathcal{S}z = z$.

Since $\mathcal{S}(\mathcal{X}) \subset \mathcal{B}(\mathcal{X})$, there exists a point $u \in \mathcal{X}$ such that $z = \mathcal{S}z = \mathcal{B}u$.

We claim that $z = \mathcal{T}u$.

For this, on putting $p = z$ and $q = u$ in (C_4), we get

$$[1+\hbar\mathfrak{H}(\mathcal{A}z,\mathcal{B}u,\mathcal{B}u)]\mathfrak{H}^2(\mathcal{S}z,\mathcal{T}u,\mathcal{T}u)$$
$$\leq \hbar\, max\left\{\begin{array}{c}\frac{1}{2}\begin{bmatrix}\mathfrak{H}^2(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{B}u,\mathcal{T}u,\mathcal{T}u)+\\ \mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}^2(\mathcal{B}u,\mathcal{T}u,\mathcal{T}u)\end{bmatrix},\\ \mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{A}z,\mathcal{T}u,\mathcal{T}u)\mathfrak{H}(\mathcal{B}u,\mathcal{S}z,\mathcal{S}z),\\ \mathfrak{H}(\mathcal{A}z,\mathcal{T}u,\mathcal{T}u)\mathfrak{H}(\mathcal{B}u,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{B}u,\mathcal{T}u,\mathcal{T}u)\end{array}\right\}$$
$$+\sigma(\mathcal{A}z,\mathcal{B}u)-\emptyset(\sigma(\mathcal{A}z,\mathcal{B}u)),$$

where
$$\sigma(\mathcal{A}z,\mathcal{B}u)=max\left\{\begin{array}{c}\mathfrak{H}^2(\mathcal{A}z,\mathcal{B}u,\mathcal{B}u),\\ \mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{B}u,\mathcal{T}u,\mathcal{T}u),\\ \mathfrak{H}(\mathcal{A}z,\mathcal{T}u,\mathcal{T}u)\mathfrak{H}(\mathcal{B}u,\mathcal{S}z,\mathcal{S}z),\\ \frac{1}{2}\begin{bmatrix}\mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{A}z,\mathcal{T}u,\mathcal{T}u)+\\ \mathfrak{H}(\mathcal{B}u,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(Bu,Tu,Tu)\end{bmatrix}\end{array}\right\}.$$

Thus we have

$$[1+\hbar\mathfrak{H}(z,z,z)]\mathfrak{H}^2(z,\mathcal{T}u,\mathcal{T}u)$$
$$\leq \hbar\, max\left\{\begin{array}{c}\frac{1}{2}\begin{bmatrix}\mathfrak{H}^2(z,z,z)\mathfrak{H}(z,\mathcal{T}u,\mathcal{T}u)+\\ \mathfrak{H}(z,z,z)\mathfrak{H}^2(z,\mathcal{T}u,\mathcal{T}u)\end{bmatrix},\\ \mathfrak{H}(z,z,z)\mathfrak{H}(z,\mathcal{T}u,\mathcal{T}u)\mathfrak{H}(z,z,z),\\ \mathfrak{H}(z,\mathcal{T}u,\mathcal{T}u)\mathfrak{H}(z,z,z)\mathfrak{H}(z,\mathcal{T}u,\mathcal{T}u)\end{array}\right\}+0-\emptyset(0).$$

Therefore,

$$[1+\hbar\mathfrak{H}(\mathcal{A}z,\mathcal{B}u,\mathcal{B}u)]\mathfrak{H}^2(\mathcal{S}z,\mathcal{T}u,\mathcal{T}u)\leq \hbar\, max\left\{\begin{array}{c}\frac{1}{2}[0+0],\\ 0,\\ 0\end{array}\right\}+0-\emptyset(0).$$

This implies that $z=\mathcal{T}u$. Since $(\mathcal{B},\mathcal{T})$ is $\mathcal{R}$-weakly commuting of type (P), we have $\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)=\mathfrak{H}(\mathcal{B}\mathcal{B}u,\mathcal{T}\mathcal{T}u,\mathcal{T}\mathcal{T}u)\leq \mathcal{R}\,\mathfrak{H}(\mathcal{T}u,\mathcal{B}u,\mathcal{B}u)=\mathcal{R}\,\mathfrak{H}(z,z,z)=0.$

Hence $\mathcal{B}z=\mathcal{T}z$.

Finally, we have

$$[1+\hbar\mathfrak{H}(\mathcal{A}z,\mathcal{B}z,\mathcal{B}z)]\mathfrak{H}^2(\mathcal{S}z,\mathcal{T}z,\mathcal{T}z)$$
$$\leq \hbar\, max\left\{\begin{array}{c}\frac{1}{2}\begin{bmatrix}\mathfrak{H}^2(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)+\\ \mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}^2(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)\end{bmatrix},\\ \mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{A}z,\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{S}z,\mathcal{S}z),\\ \mathfrak{H}(\mathcal{A}z,\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)\end{array}\right\}$$
$$+\sigma(\mathcal{A}z,\mathcal{B}z)-\emptyset(\sigma(\mathcal{A}z,\mathcal{B}z)),$$

where $\sigma(\mathcal{A}z,\mathcal{B}z) = max\left\{\begin{matrix} \mathfrak{H}^2(\mathcal{A}z,\mathcal{B}z,\mathcal{B}z), \\ \mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z), \\ \mathfrak{H}(\mathcal{A}z,\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{S}z,\mathcal{S}z), \\ \frac{1}{2}\begin{bmatrix}\mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z)d(\mathcal{A}z,\mathcal{T}z,\mathcal{T}z) + \\ \mathfrak{H}(\mathcal{B}z,\mathcal{S}z,\mathcal{S}z)\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)\end{bmatrix} \end{matrix}\right\} = \mathfrak{H}^2(z,\mathcal{B}z,\mathcal{B}z).$

On simplification, we have

$$[1 + \hbar\mathfrak{H}(z,\mathcal{T}z,\mathcal{T}z)]\mathfrak{H}^2(z,\mathcal{T}z,\mathcal{T}z) \le \hbar\, max\left\{\begin{matrix}\frac{1}{2}[0+0], \\ 0, \\ 0\end{matrix}\right\} + \mathfrak{H}^2(z,\mathcal{T}z,\mathcal{T}z) - \emptyset(\mathfrak{H}^2(z,\mathcal{T}z,\mathcal{T}z)).$$

This implies that $z = \mathcal{T}z$. Hence $z = \mathcal{B}z = \mathcal{T}z = \mathcal{A}z = \mathcal{S}z$. Therefore, z is a common fixed point of $\mathcal{S}, \mathcal{T}, \mathcal{A}$and $\mathcal{B}$.

Case 2: Suppose that $\mathcal{B}$ is continuous. Then we can obtain the same result by using Case 1.

Case 3: Suppose that $\mathcal{S}$ is continuous.

Then $\{\mathcal{S}\mathcal{S}x_{2n}\}$and $\{\mathcal{S}\mathcal{A}x_{2n}\}$ converge to $\mathcal{S}z$ as $n \to \infty$.

Since the mappings $\mathcal{A}$ and $\mathcal{S}$ are $\mathcal{R}$-weakly commuting of type (P), we have $\mathfrak{H}(\mathcal{A}\mathcal{A}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n}) \le \mathcal{R}\,\mathfrak{H}(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})$.

Letting $n \to \infty$, we get $\lim_{n\to\infty} \mathcal{A}\mathcal{S}\,x_{2n} = \mathcal{S}z$.

On putting$p = \mathcal{S}x_{2n}$ and $q = x_{2n+1}$ in (C_4), we get

$$[1 + \hbar\mathfrak{H}(\mathcal{A}\mathcal{S}x_{2n},\mathcal{B}x_{2n+1},\mathcal{B}x_{2n+1})]\mathfrak{H}^2(\mathcal{S}\mathcal{S}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})$$
$$\le \hbar\, max\left\{\begin{matrix} \frac{1}{2}\begin{bmatrix}\mathfrak{H}^2(\mathcal{A}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}) \\ +\mathfrak{H}(\mathcal{A}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})\mathfrak{H}^2(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\end{bmatrix}, \\ \mathfrak{H}(\mathcal{A}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{A}\mathcal{S}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n}), \\ \mathfrak{H}(\mathcal{A}\mathcal{S}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}) \end{matrix}\right\}$$
$$+\sigma(\mathcal{A}\mathcal{S}x_{2n},\mathcal{B}x_{2n+1}) - \emptyset(\sigma(\mathcal{A}\mathcal{S}x_{2n},\mathcal{B}x_{2n+1})),$$

where

$$\sigma(\mathcal{A}\mathcal{S}x_{2n},\mathcal{B}x_{2n+1})$$
$$= max\left\{\begin{matrix} \mathfrak{H}^2(\mathcal{A}\mathcal{S}x_{2n},\mathcal{B}x_{2n+1},\mathcal{B}x_{2n+1}), \mathfrak{H}(\mathcal{A}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}), \\ \mathfrak{H}(\mathcal{A}\mathcal{S}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n}), \\ \frac{1}{2}\begin{bmatrix}\mathfrak{H}(\mathcal{A}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{A}\mathcal{S}x_{2n},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1}) \\ +\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{S}\mathcal{S}x_{2n},\mathcal{S}\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}x_{2n+1},\mathcal{T}x_{2n+1},\mathcal{T}x_{2n+1})\end{bmatrix} \end{matrix}\right\}.$$

Letting$n$ $n \to \infty$,we have

$$[1 + \hbar\mathfrak{H}(\mathcal{S}z, z, z)]\mathfrak{H}^2(\mathcal{S}z, z, z) \le \hbar\, max \begin{Bmatrix} \frac{1}{2}[0 + 0], \\ 0, \\ 0 \end{Bmatrix} + \mathfrak{H}^2(\mathcal{S}z, z, z) - \emptyset(\mathfrak{H}^2(\mathcal{S}z, z, z)),$$

i.e., $[1 + \hbar\mathfrak{H}(\mathcal{S}z, z, z)]\mathfrak{H}^2(\mathcal{S}z, z, z) \le \mathfrak{H}^2(\mathcal{S}z, z, z) - \emptyset(\mathfrak{H}^2(\mathcal{S}z, z, z))$,

Thus we get $\mathfrak{H}^2(\mathcal{S}z, z, z) = 0$, which implies that $\mathcal{S}z = z$.

Since $\mathcal{S}(\mathcal{X}) \subset \mathcal{B}(\mathcal{X})$, there exists a point $v \in \mathcal{X}$ such that $z = \mathcal{S}z = \mathcal{B}v$.

We claim that $z = \mathcal{T}v$.

For this, putting $p = \mathcal{S}x_{2n}$ and $q = v$ in (C_4), we get

$$[1 + \hbar\mathfrak{H}(\mathcal{AS}x_{2n}, \mathcal{B}v, \mathcal{B}v)]\mathfrak{H}^2(\mathcal{SS}x_{2n}, \mathcal{T}v, \mathcal{T}v)$$
$$\le \hbar\, max \begin{Bmatrix} \frac{1}{2}\begin{bmatrix} \mathfrak{H}^2(\mathcal{AS}x_{2n}, \mathcal{SS}x_{2n}, \mathcal{SS}x_{2n})\mathfrak{H}(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v) \\ +\mathfrak{H}(\mathcal{AS}x_{2n}, \mathcal{SS}x_{2n}, \mathcal{SS}x_{2n})\mathfrak{H}^2(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v) \end{bmatrix}, \\ \mathfrak{H}(\mathcal{AS}x_{2n}, \mathcal{SS}x_{2n}, \mathcal{SS}x_{2n})\mathfrak{H}(\mathcal{AS}x_{2n}, \mathcal{T}v, \mathcal{T}v)\mathfrak{H}(\mathcal{B}v, \mathcal{SS}x_{2n}, \mathcal{SS}x_{2n}), \\ \mathfrak{H}(\mathcal{AS}x_{2n}, \mathcal{T}v, \mathcal{T}v)\mathfrak{H}(\mathcal{B}v, \mathcal{SS}x_{2n}, \mathcal{SS}x_{2n})\mathfrak{H}(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v) \end{Bmatrix}$$
$$+\sigma(\mathcal{AS}x_{2n}, \mathcal{B}v) - \emptyset(\sigma(\mathcal{AS}x_{2n}, \mathcal{B}v)),$$

where $\sigma(\mathcal{AS}x_{2n}, \mathcal{B}v) = max \begin{Bmatrix} \mathfrak{H}^2(\mathcal{AS}x_{2n}, \mathcal{B}v, \mathcal{B}v), \\ \mathfrak{H}(\mathcal{AS}x_{2n}, \mathcal{SS}x_{2n}, \mathcal{SS}x_{2n})\mathfrak{H}(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v), \\ \mathfrak{H}(\mathcal{AS}x_{2n}, \mathcal{T}v, \mathcal{T}v)\mathfrak{H}(\mathcal{B}v, \mathcal{SS}x_{2n}, \mathcal{SS}x_{2n}), \\ \frac{1}{2}\begin{bmatrix} \mathfrak{H}(\mathcal{AS}x_{2n}, \mathcal{SS}x_{2n}, \mathcal{SS}x_{2n})\mathfrak{H}(\mathcal{AS}x_{2n}, \mathcal{T}v, \mathcal{T}v) + \\ \mathfrak{H}(\mathcal{B}v, \mathcal{SS}x_{2n}, \mathcal{SS}x_{2n})\mathfrak{H}(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v) \end{bmatrix} \end{Bmatrix}.$

On simplification, we get

$$[1 + \hbar\, \mathfrak{H}(z, z, z)]\mathfrak{H}^2(z, \mathcal{T}v, \mathcal{T}v)$$
$$\le \hbar\, max \begin{Bmatrix} \frac{1}{2}[\mathfrak{H}^2(z, z, z)\mathfrak{H}(z, \mathcal{T}v, \mathcal{T}v) + \mathfrak{H}(z, z, z)\mathfrak{H}^2(z, \mathcal{T}v, \mathcal{T}v)], \\ \mathfrak{H}(z, z, z)\mathfrak{H}(z, \mathcal{T}v, \mathcal{T}v)\mathfrak{H}(z, z, z), \\ \mathfrak{H}(z, \mathcal{T}v, \mathcal{T}v)\mathfrak{H}(z, z, z)\mathfrak{H}(z, \mathcal{T}v, \mathcal{T}v) \end{Bmatrix} + 0 - \emptyset(0).$$

This implies that $z = \mathcal{T}v$. Since $(\mathcal{B}, \mathcal{T})$ is $\mathcal{R}$ −weakly commuting of type (P), we have

$$\mathfrak{H}(\mathcal{T}z, \mathcal{B}z, \mathcal{B}z) = \mathfrak{H}(\mathcal{TT}v, \mathcal{BB}v, \mathcal{BB}v) \le \mathcal{R}\, \mathfrak{H}(\mathcal{B}v, \mathcal{T}v, \mathcal{T}v) = \mathcal{R}\, \mathfrak{H}(z, z, z) = 0.$$

This gives $\mathcal{B}z = \mathcal{T}z$, for $\mathcal{R} > 0$.

Finally, from (C_4) we have

$$[1+\hbar\,\mathfrak{H}(\mathcal{A}x_{2n},\mathcal{B}z,\mathcal{B}z)]\mathfrak{H}^2(\mathcal{S}x_{2n},\mathcal{T}z,\mathcal{T}z)$$

$$\leq \hbar\, max\left\{\begin{array}{c}\frac{1}{2}\begin{bmatrix}\mathfrak{H}^2(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)+\\ \mathfrak{H}(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}^2(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)\end{bmatrix},\\ \mathfrak{H}(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{A}x_{2n},\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{S}x_{2n},\mathcal{S}x_{2n}),\\ \mathfrak{H}(\mathcal{A}x_{2n},\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)\end{array}\right\}$$

$$+\sigma(\mathcal{A}x_{2n},\mathcal{B}z)-\emptyset(\sigma(\mathcal{A}x_{2n},\mathcal{B}z)),$$

where $\sigma(\mathcal{A}x_{2n},\mathcal{B}z)=max\left\{\begin{array}{c}\mathfrak{H}^2(\mathcal{A}x_{2n},\mathcal{B}z,\mathcal{B}z),\\ \mathfrak{H}(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z),\\ \mathfrak{H}(\mathcal{A}x_{2n},\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{S}x_{2n},\mathcal{S}x_{2n}),\\ \frac{1}{2}\begin{bmatrix}\mathfrak{H}(\mathcal{A}x_{2n},\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{A}x_{2n},\mathcal{T}z,\mathcal{T}z)+\\ \mathfrak{H}(\mathcal{B}z,\mathcal{S}x_{2n},\mathcal{S}x_{2n})\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)\end{bmatrix}\end{array}\right\}.$

Therefore, we have

$$[1+\hbar\mathfrak{H}(z,\mathcal{T}z,\mathcal{T}z)]\mathfrak{H}^2(z,\mathcal{T}z,\mathcal{T}z)$$

$$\leq \hbar\, max\left\{\begin{array}{c}\frac{1}{2}[0+0],\\ 0,\\ 0\end{array}\right\}+\mathfrak{H}^2(z,\mathcal{T}z,\mathcal{T}z)-\emptyset(\mathfrak{H}^2(z,\mathcal{T}z,\mathcal{T}z)).$$

This gives $z=\mathcal{T}z$. Since $\mathcal{T}(\mathcal{X})\subset\mathcal{A}(\mathcal{X})$, therefore, there exists a point $w\in\mathcal{X}$ such that $z=\mathcal{T}z=\mathcal{A}w$.

We claim that $z=\mathcal{S}w$.For this, putting $p=w$ and $q=z$ in (C_4), we get

$$[1+\hbar\,\mathfrak{H}(\mathcal{A}w,\mathcal{B}z,\mathcal{B}z)]\mathfrak{H}^2(\mathcal{S}w,\mathcal{T}z,\mathcal{T}z)$$

$$\leq \hbar\, max\left\{\begin{array}{c}\frac{1}{2}\begin{bmatrix}\mathfrak{H}^2(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)+\\ \mathfrak{H}(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}^2(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)\end{bmatrix},\\ \mathfrak{H}(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(\mathcal{A}w,\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{S}w,\mathcal{S}w),\\ \mathfrak{H}(\mathcal{A}w,\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)\end{array}\right\}$$

$$+\sigma(\mathcal{A}w,\mathcal{B}z)-\emptyset(\sigma(\mathcal{A}w,\mathcal{B}z)),$$

where $\sigma(\mathcal{A}w,\mathcal{B}z)=max\left\{\begin{array}{c}\mathfrak{H}^2(\mathcal{A}w,\mathcal{B}z,\mathcal{B}z),\\ \mathfrak{H}(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z),\\ \mathfrak{H}(\mathcal{A}w,\mathcal{T}z,\mathcal{T}z)\mathfrak{H}(\mathcal{B}z,\mathcal{S}w,\mathcal{S}w),\\ \frac{1}{2}\begin{bmatrix}\mathfrak{H}(\mathcal{A}w,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(\mathcal{A}w,\mathcal{T}z,\mathcal{T}z)+\\ \mathfrak{H}(\mathcal{B}z,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(\mathcal{B}z,\mathcal{T}z,\mathcal{T}z)\end{bmatrix}\end{array}\right\},$

i.e.,$\sigma(\mathcal{A}w,\mathcal{B}z)=max\left\{\begin{array}{c}\mathfrak{H}^2(z,z,z),\\ \mathfrak{H}(z,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(\mathcal{T}z,\mathcal{T}z,\mathcal{T}z),\\ \mathfrak{H}(z,z,z)\mathfrak{H}(z,\mathcal{S}w,\mathcal{S}w),\\ \frac{1}{2}\begin{bmatrix}\mathfrak{H}(z,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(z,z,z)+\\ \mathfrak{H}(z,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(\mathcal{T}z,\mathcal{T}z,\mathcal{T}z)\end{bmatrix}\end{array}\right\}=0.$

Hence we get $[1 + \hbar\mathfrak{H}(z,z,z)]\mathfrak{H}^2(\mathcal{S}w, z, z)$

$$\leq \hbar\, max\left\{\begin{array}{c}\frac{1}{2}\begin{bmatrix}\mathfrak{H}^2(z,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(z,z,z) + \\ \mathfrak{H}(z,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}^2(z,z,z)\end{bmatrix}, \\ \mathfrak{H}(z,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(z,z,z)\mathfrak{H}(z,\mathcal{S}w,\mathcal{S}w), \\ \mathfrak{H}(z,z,z)\mathfrak{H}(z,\mathcal{S}w,\mathcal{S}w)\mathfrak{H}(z,z,z)\end{array}\right\} + 0 - \emptyset(0),$$

which implies that $\mathcal{S}w = z$. Since $(\mathcal{S},\mathcal{A})$ is $\mathcal{R}$-weakly commuting of type (P), we have $\mathfrak{H}(\mathcal{A}z,\mathcal{S}z,\mathcal{S}z) = \mathfrak{H}(\mathcal{A}\mathcal{A}w,\mathcal{S}\mathcal{S}w,\mathcal{S}\mathcal{S}w) \leq \mathcal{R}\,\mathfrak{H}(\mathcal{S}w,\mathcal{A}w,\mathcal{A}w) = \mathcal{R}\,\mathfrak{H}(z,z,z) = 0$. Hence $\mathcal{A}z = \mathcal{S}z$. Hence $z = \mathcal{A}z = \mathcal{S}z = \mathcal{B}z = \mathcal{T}z$, and z is a common fixed point of $\mathcal{S},\mathcal{T},\mathcal{A}$and $\mathcal{B}$.

Case 4: Suppose that $\mathcal{T}$ is continuous. We can obtain the same result by using Case 3.

Uniqueness: Suppose that $z \neq w$ are two common fixed points of $\mathcal{S},\mathcal{T},\mathcal{A}$and $\mathcal{B}$.

On putting $p = z$ and $q = w$ in (C_4), we get

$$[1 + \hbar\,\mathfrak{H}(\mathcal{A}z,\mathcal{B}w,\mathcal{B}w)]\mathfrak{H}^2(\mathcal{S}z,\mathcal{T}w,\mathcal{T}w)$$
$$\leq \hbar\, max\{0,0,0\} + \sigma(\mathcal{A}z,\mathcal{B}w) - \emptyset(\sigma(\mathcal{A}z,\mathcal{B}w))$$

$i.e., \mathfrak{H}^2(z,w,w) = 0$ implies $z = w$. This completes the proof.

Example 3.1 Let $\mathcal{X} = [6, 24]$ and $\mathfrak{H}$ be a usual $\mathfrak{H}-$ metric space defined by $\mathfrak{H}(x,y,z) = |x-y| + |y-z| + |z-x|$ for all $x,y,z \in \mathcal{X}$. Define the self-mappings $\mathcal{S},\mathcal{T},\mathcal{A}$and $\mathcal{B}$ on $\mathcal{X}$ by

$$\mathcal{A}x = \left\{\begin{array}{ll}16 & if\ \ 6 < x \leq 9 \\ x-3 & if\ \ \ x > 9 \\ 6 & if\ \ \ \ x = 6\end{array}\right\}; \qquad \mathcal{B}x = \left\{\begin{array}{ll}6 & if\ \ x = 6 \\ 10 & if\ x > 6\end{array}\right\};$$

$$\mathcal{S}x = \left\{\begin{array}{ll}10 & if\ \ 6 < x \leq 9 \\ x & if\ x = 6 \\ 6 & if\ x > 9\end{array}\right\}; \qquad \mathcal{T}x = \left\{\begin{array}{ll}x & if\ x = 6 \\ 7 & if\ x > 6\end{array}\right\}.$$

Let us consider a $\{x_n\}$ with $x_n = 6$. All the conditions of theorem 4.1 are satisfied. Thus 6 is unique common fixed point of $\mathcal{S},\mathcal{T},\mathcal{A}$ and $\mathcal{B}$.

Conclusion In this paper, we prove common fixed point theorems for variants of R-weakly commuting and reciprocal mappings in $\mathfrak{H}$ -metric space that contains cubic and quadratic terms of distance function $\mathfrak{H}(x,y,z)$.

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References

[1] Alber Y.I. and Delabriere S. G., Principle of weakly contractive maps in Hilbert spaces, *Operator Theory Advances and Applications*, **98** (1997), 7-22.

[2] Boyd D. W. and Wong J. S. W., *On nonlinear contractions*, *Proc. Amer. Math. Soc*., **20(2)** (1969), 458–464,.

[3] Jungck G. and Rhoades B. E., Fixed points for set valued functions without continuity, *Indian J. Pure Appl. Math.,* **29(3)** (1998), 227-238.

[4] Jungck G., Common Fixed Points for Non continuous Non self Maps on Non metric Spaces, *Far East J. Math. Sci.,* **4(2)**(1996), 199- 215.

[5] Jungck G., Commuting mappings and fixed points, *Amer. Math. Monthly*, **83**(1976), 261-263.

[6] Jungck G., Compatible mappings and common fixed points (2), *Int. J. Math. and Math. Sci.,* **11**(1988), 285-288.

[7] Jungck G., Compatible mappings and common fixed points, *Int. J. Math. Math. Sci.,* **9** (1986), 771-779.

[8] Manro S., Bhatia S.S., Kumar S., Expansion mapping theorems in G-metric spaces, *Int. J. Contemp. Math. Sci.,***5 (51)** (2010), 2529-2535.

[9] Murthy P.P. and Prasad K.N.V.V.V., Weak contraction condition involving cubic terms of d(x, y) under the fixed point consideration, *J. Math.,* Article ID 967045, 5 pages. doi: 10.1155/2013/967045.

[10] Mustafa Z. and Sims B., A new approach to generalized metric spaces, *Journal of Nonlinear and Convex Analysis,***7(2)** (2006), 289-297.

[11] Mustafa Z. and Sims B., Sims, Fixed point theorems for contractive mappings in G-metric spaces, *Fixed Point Theory and Applications*, vol. 2009, Article ID 917175.

[12] Mustafa Z. and Sims B., Some remarks Concerning D-metric spaces, Proceedings of International Conference on Fixed Point Theory and Applications, pp.189-198, Yokohama, Japan, 2004.

[13] Mustafa Z., Obiedat H. and Awawdeh M., Some fixed point theorem for mapping on complete G-metric spaces, *Fixed Point Theory and Applications*, vol. 2008, Article ID 189870.

[14] Mustafa Z., Shatanawi W. and Bataineh M., Fixed point theorem on un complete G-metric spaces, *Journal of Mathematics and Statistics*, **4**(2008), 196-201.

[15] Mustafa Z., Shatanawi W. and Bataineh M.,, Existence of fixed point results in G-metric spaces, *Int. J. of Math. and Math. Sci*., vol. 2009, Article ID 283028.

[16] Pant R. P., Common fixed point for contractive maps, *J. Math. Anal. Appl.,***226** (1998), 251-258.

[17] Pant R. P., Common fixed point for non-commuting mappings, *J. Math. Anal. Appl.,***188** (1994), 436-440.

[18] Pant R. P., Common fixed points of four mappings, *Bull. Calcutta Math. Soc.,***90** (1998) 281-286.

[19] Pant R. P., R-weak Commutativity and common fixed points, *Soochow J. Math.,***49** (1999) 37-42.

[20] Rhoades B. E., Some theorems on weakly contractive maps, *Nonlinear Analysis: Theory, Methods & Applications,* **47(4)** (2001), 2683–2693.

[21] Sessa S., On a weak commutativity conditions of mappings in fixed point consideration, *Publ. Inst. Math. Beograd*, **32(46)** (1982), 146-153.

[22] Vats R.K., Kumar S., Sinag V., Some common fixed point theorem for compatible mappings of type(A) in complete G-metric space, *Adv. Fuzzy Math.,* **6 (1)** (2011), 27-38.