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Every Tree is a Subtree of a k-Equitable Tree for any $k \geq 2$

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Abstract

As the Graceful Tree Conjecture and the characterization of graceful graphs are extremely difficult problems to settle, different possible generalization of graceful labeling were introduced for a better understanding of the above two problems. One such generalization of graceful labeling is called k-equitable labeling, which was introduced by Cahit [2] in 1990. For a graph $G(V, E)$ and for a positive integer $k \geq 2$, a function f defined from the vertex set of G to $\{0, 1, 2, \ldots, k-1\}$ is called k-equitable if every edge uv is assigned the label $|f(u) - f(v)|$, then the number of vertices labeled i and the number of vertices labeled j differ by at most 1 and the number of edges labeled i and the number of edges labeled j differ by at most 1, for i, j, $0 \leq i < j \leq k-1$. Note that a graph with m edges, is graceful if and only if it is $m+1$ equitable. In 1990 Cahit [2] conjectured that every tree is k-equitable for any $k \geq 2$. This conjecture is equivalent to the celebrated Graceful Tree Conjecture when k is the number of vertices of the tree. The result of Niall Cairnie and Keith Edward [7] imply that the recognition of whether the graph is k-equitable for $k \geq 2$ is an NP-complete problem. The Cahit's k-Equitable Tree Conjecture and its relevance to the Graceful Tree Conjecture and the result that the recognition of k equitable graph is NP-complete strongly motivated to understand the structure of k equitable graphs. In this direction here in this paper we prove a fundamental structural property of k-equitable trees that every tree is k-equitable.

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1. INTRODUCTION

In 1963, Ringel [8] conjectured that K_{2m+1}, the complete graph on $2m+1$ vertices, can be decomposed into $2m+1$ isomorphic copies of a given tree with m-edges. In 1965, Kotzig [6] conjectured that the complete graph K_{2m+1}, can be cyclically decomposed into $2m+1$ copies of a given tree with m edges. To settle the above two conjectures, in 1967, Rosa [9] introduced a hierarchical series of labeling called ρ, σ, β and α-valuations. Later, Golomb [4] called β-valuation as graceful and now this is the term most widely used.

A graceful labeling of a graph G with m edges and vertex set V is an injection $f : V(G) \to \{0, 1, 2, \ldots, m\}$ with the property that the resulting edge labels are also distinct where an edge incident with vertices u and v is assigned the label $|f(u) - f(v)|$. A graph which admits a graceful labeling is called a graceful graph. In his classical paper [9], Rosa proved the theorem that, if a tree T with m edges has a decomposition into $2m+1$ copies of T.

From Rosa's theorem it follows that both Ringel and Kotzig's conjectures are true if every tree is graceful. This led to the birth of the popular Ringel-Kotzig-Rosa conjecture popularly called the Graceful Tree Conjecture: "All trees are graceful".

As the Graceful Tree Conjecture is a hard problem to settle, and the characterization of graceful graphs are extremely hard to understand, different generalization on graceful labeling were introduced and studied. One such generalization of graceful labeling is a k-equitable labeling, which was introduced by Cahit [2] in the year 1990. In the k-equitable labeling, the vertex and the edge labels are distributed as evenly as possible and it is defined more precisely in the following way.

For a graph $G(V, E)$ and for a positive integer $k \geq 2$, a function $f : V(G) \to \{0, 1, 2, \ldots, k-1\}$ is called a k-equitable labeling, if f and its induced edge labeling function $f^* : E(G) \to \{0, 1, 2, \ldots, k-1\}$ defined by $f^*(e = uv) = |f(u) - f(v)|$ satisfying the condition $|v_f(i) - v_f(j)| \leq 1$ and $|e_f(i) - e_{f^*}(j)| \leq 1$ respectively, for i, j, $0 \leq i < j \leq k-1$ where $v_f(i)$ and $e_f(i)$ denote the number of vertices and the number of edges having the label i under f and f^* respectively.

Note that a graph with m edges, is graceful if and only if it is $m+1$ equitable. In 1990 Cahit [2] conjectured that every tree is k-equitable for any $k \geq 2$. This conjecture is equivalent to the celebrated Graceful Tree Conjecture when k is the number of vertices of the tree.

Cahit [1] proved that every tree is 2-equitable. The 2-equitable labeling is called popularly as cordial labeling. Speyer and Szaniszlo [11] proved that every tree is

3-equitable. Szaniszlo [10] proved that every path is k-equitable and every star is k-equitable. For an exhaustive survey on k-equitable graph refer the excellent dynamic survey by Gallian [5].

In 2000 Niall Cairnie and Keith Edward [7] proved that the recognition of whether the graph is k-equitable for $k \geq 2$ is an NP-complete problem. The Cahit's k-Equitable Tree Conjecture and its relevance to the Graceful Tree Conjecture and the result that the recognition of k equitable graph is NP-complete strongly motivated to understand the structure of k equitable graphs. In this direction here in this paper we prove a fundamental structural property of k-equitable trees that every tree is k-equitable.

2. MAIN RESULT

In this section we prove our main result in Theorem 2.1. As our main result uses the Horse-Race Labeling Algorithm proposed by Cahit [1]. We present the Horse-Race Labeling Algorithm for the completeness.

Horse-Race Labeling Algorithm

1. Choose $v_1 \in V(T)$; set $f(v_1) = 0$, $V_L = \{v_1\}$, $i = 1$, $v_L(0) = 1$, $v_L(1) = e_L(1) = e_L(0) = 0$.
2. If $V \backslash V_L = \phi$, stop. Otherwise, replace i by $i + 1$.
3. Choose $v \in V \backslash V_L$ adjacent to $u \in V_L$, and let $v_i \in V$.

   If $v_L(1) > v_L(0)$ put $f(v_i) = 0$.

   If $v_L(1) < v_L(0)$ put $f(v_i) = 1$.

   If $v_L(1) = v_L(0)$ and $e_L(1) > e_L(0)$ put $f(v_i) = f(u)$.

   If $v_L(1) = v_L(0)$ and $e_L(1) < e_L(0)$ put $f(v_i) = 1 - f(u)$.

   Replace V_L by $V_L \cup \{v_i\}$, update $v_L(j)$, $e_L(j)$, and go to Step 2.

Theorem 2.1. *For each $k \geq 2$, every tree is a subtree of a k-equitable tree.*

Proof.

Step 1. Construction of a tree T^* containing a given arbitrary tree T as its subtree

Consider an arbitrary tree T. Let $|V(T)| = N$. Consider the set $S = \{y_1, y_2, \ldots, y_{(k-2)(\lfloor \frac{N}{2} \rfloor)}\}$ of new vertices. Then select r vertices,

$1 \leq r \leq N$, say $x_1, x_2, \dots, x_r$ from the tree T. Now, for each x_i, $1 \leq i \leq r$, choose s_i distinct vertices when k is odd while when k is even, choose s_i distinct vertices where s_i is an even number. Let the chosen s_i vertices from the set S to be $y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_{s_i}}$, where $\alpha_j \in \{1, 2, \dots, (k-2)\lfloor \frac{N}{2} \rfloor\}$ with $1 \leq s_i \leq (k-2)(\lfloor \frac{N}{2} \rfloor)$ and $s_1 + s_2 + \cdots + s_r = (k-2)(\lfloor \frac{N}{2} \rfloor)$. Then join each x_i, for i, $1 \leq i \leq r$ with its chosen new vertices $y_{\alpha_1}, y_{\alpha_2}, \dots, y_{\alpha_{s_i}}$, denote the resultant graph by T^*. As $x_i y_{\alpha_j}$, for each j, $1 \leq j \leq s_i$ is a distinct pendant edge in T^*, the graph T^* is also a tree containing the arbitrary tree T.

Step 2. Defining a k-equitable labeling of T^*

Step 2.1. Labeling the vertices of T

Consider the given arbitrary tree T (which is a subtree) in the tree T^*. Then, using the Horse-Race Labeling Algorithm, obtain the 0-1 cordial labeling of the vertices of T. Then relabel every vertex of T having the label 1 with the label $k-1$.

Let $V_0(T)$ denotes the set of all vertices of T having the label 0 and let $V_{k-1}(T)$ denotes the set of all vertices of T having the label $k-1$. As the Horse-Race Labeling Algorithm generates cordial labeling on T, we have

$$||V_0(T)| - |V_{k-1}(T)|| \leq 1 \text{ and} \tag{1}$$

$$||E_0(T)| - |E_{k-1}(T)|| \leq 1 \tag{2}$$

where $E_0(T)$ denotes the set of all edges of T having the label 0 and $E_{k-1}(T)$ denotes the set of all edges of T having the label $k-1$.

Step 2.2. Extending the labeling of T to T^*

We extend the labeling of T to a k-equitable labeling of the tree T^* by assigning the labels $1, 2, \dots, k-2$ to the vertices of S in T^*.

Consider the set S of new vertices added to the tree T in T^* and let S_1 be the subset of vertices in S which are adjacent to the vertices of T labeled with 0, and let S_2 be the subset of vertices in S which are adjacent to the vertices of T labeled with $k-1$. It is clear that (S_1, S_2) is a partition of the set S.

Let S'_1 be the set of all edges of T^* having one end in S_1 and the other end in T with label 0. Similarly, let S'_2 be the set of all edges of T^* having one end in S_2 and the other end in T with label $k-1$. To define a k-equitable labeling for T^*, we will assign

$\lfloor \frac{N}{2} \rfloor$ times each of the labels $1, 2, \ldots, k-2$ to the vertices in the set $S_1 \cup S_2 = S$. Let $|S_1| = m$ and $|S_2| = n$. We can write

$$m = a(k-2) + b, \text{ where } 0 \leq b < k-2 \quad \text{and} \tag{3}$$

$$n = c(k-2) + d, \text{ where } 0 \leq d < k-2. \tag{4}$$

Claim 1: Either $b + d = 0$ with $b = 0$ and $d = 0$ or $b + d = k - 2$ with $b > 0$, $d > 0$

We have $m + n = |S_1| + |S_2| = |S| = (k-2)\left(\lfloor \frac{N}{2} \rfloor\right)$. From (3) and (4) we have $a(k-2) + b + c(k-2) + d = (k-2)\left(\lfloor \frac{N}{2} \rfloor\right)$. That is, $b + d = (k-2)\left(\lfloor \frac{N}{2} \rfloor - (a+c)\right)$. Therefore, $b + d$ is a multiple of $k - 2$. Since $b + d$ is a multiple of $k-2$ and $0 \leq b + d < 2(k-2)$, we must have either $b + d = 0$ or $b + d = k - 2$. If $b + d = k - 2$, then $b \neq 0$. If $b = 0$, then $d = k - 2$, but $0 \leq d < k - 2$, a contradiction. Also, if $b + d = k - 2$, then $d \neq 0$. If $d = 0$, then $b = k - 2$, but $0 \leq b < k - 2$, a contradiction. Therefore, $b > 0$ and $d > 0$ when $b + d = k - 2$. From (3) and (4), we have $b \geq 0$ and $d \geq 0$. Then, if $b + d = 0$, implies $b = 0 = d$. Hence the claim.

The assignment of the labels $1, 2, \ldots, k-2$ to the vertices of S in T^* depends on the cases $b + d = 0$ with $b = d = 0$ or the case $b + d = k - 2$ with $b > 0$ and $d > 0$.

Step 2.2.1. Labeling the vertices of S in T^* when $b + d = 0$ with $b = d = 0$

Since $b = d = 0$, we have from (3) and (4) $m = a(k-2)$, $n = c(k-2)$

Then identify a distinct subsets of S_1 each subset containing $k - 2$ (unlabeled pendant) vertices, and for each such subset having $k-2$ unlabeled pendant vertices, respectively assign the labels $1, 2, \ldots, k-2$. Similarly, identify c distinct subsets of S_2 each such subset having $k - 2$ (unlabeled pendant) vertices. Then for each such subset having $k - 2$ unlabeled pendant vertices, respectively assign the labels $1, 2, \ldots, k-2$.

Let $V_i(T^*)$ denote the set of all vertices of T^* having the label i, for i, $0 \leq i \leq k-1$, and $E_i(T^*)$ denote the set of all edges of T^* having the label i, for i, $0 \leq i \leq k-1$.

Then by the above labeling, we have

$$|V_i(T^*)| = |V_j(T^*)| \text{ for } 1 \leq i < j \leq k-2 \tag{5}$$

$$|E_i(T^*)| = |E_j(T^*)| \text{ for } 1 \leq i < j \leq k-2 \tag{6}$$

As T is a subtree of T^* and the labels 0 and $k-1$ are not assigned to any of the vertices in S of T^*.

$$|V_0(T^*)| = |V_0(T)| \text{ and } |V_{k-1}(T^*)| = |V_{k-1}(T)| \text{ and}$$
$$|E_0(T^*)| = |E_0(T)| \text{ and } |E_{k-1}(T^*)| = |E_{k-1}(T)|$$

We have, from (1) and (2)

$$||V_0(T)| - |V_{k-1}(T)|| \leq 1 \quad \text{and} \quad ||E_0(T)| - |E_{k-1}(T)|| \leq 1$$

From (1), (2), (5) and (6), we have

$$||V_i(T^*)| - |V_j(T^*)|| \leq 1$$
$$||E_i(T^*)| - |E_j(T^*)|| \leq 1 \text{ for } i, j, 0 \leq i < j \leq k-1.$$

Thus, T^* is k-equitable.

Step 2.2.2. Labeling of the vertices of S in T^* when $b + d = k - 2$ with $b > 0$ and $d > 0$,

From (3) and (4), we have $m = a(k-2)+b$, where $0 \leq b < k-2$ and $n = c(k-2)+d$, where $0 \leq d < k-2$. $m+n = (a+c)(k-2)+(b+d)$. For the $(a+c)(k-2)$ vertices, we label them as in the Step 2.2.1. For the remaining $b+d$ vertices, we consider two cases depending on the nature of k is either even or odd.

Case 1. k is odd

Then $k-2$ is odd. As $b+d = k-2$, we have $b+d =$ odd. This would imply that either b is odd and d is even or b is even and d is odd. Then to extend the k-equitable labeling of T to T^*, we further consider the following subcases.

Case 1.1. b is odd and d is even

Then we can write $b = 2z+1$ for some $z \in Z^+$. Then assign the b labels given in the following set A to the b vertices of T^* in S_1.

$$A = \left\{ \left(\frac{k-1}{2}\right) - z, \left(\frac{k-1}{2}\right) - (z-1), \left(\frac{k-1}{2}\right) - (z-2), \ldots, \right.$$
$$\left. \left(\frac{k-1}{2}\right) + (z-2), \left(\frac{k-1}{2}\right) + (z-1), \left(\frac{k-1}{2}\right) + z \right\}$$

Then the induced edge labels of the pendant edges having one end vertex in S_1 and the other end vertex with the label 0 in T contained in T^* are given in the following set A'.

$$A' = \left\{ \left(\frac{k-1}{2}\right) - z, \left(\frac{k-1}{2}\right) - (z-1), \left(\frac{k-1}{2}\right) - (z-2), \right.$$
$$\left. \ldots, \left(\frac{k-1}{2}\right) + (r-2), \left(\frac{k-1}{2}\right) + (z-1), \left(\frac{k-1}{2}\right) + z \right\}$$

Now assign the d labels given in the following set B to the d vertices of T^* in S_2.

$B = \{1, 2, \ldots, k-3, k-2\} \backslash A$. That is,

$$B = \left\{ 1, 2, 3, \ldots, \left(\frac{k-1}{2}\right) - (z+3), \left(\frac{k-1}{2}\right) - (z+2), \right.$$
$$\left. \left(\frac{k-1}{2}\right) - (z+1) \right\} \cup \left\{ \left(\frac{k-1}{2}\right) + (z+1), \left(\frac{k-1}{2}\right) + (z+2), \right.$$
$$\left. \left(\frac{k-1}{2}\right) + (z+3), \ldots, k-4, k-3, k-2 \right\}$$

Then the induced edge labels of the pendant edges having one end vertex in S_2 and the other end vertex with the label $k-1$ in T, contained in T^* are given in the following set B'.

$$B' = \left\{ k-2, k-3, k-4, \ldots, \left(\frac{k-1}{2}\right) + (z+3), \left(\frac{k-1}{2}\right) + (z+2), \right.$$
$$\left. \left(\frac{k-1}{2}\right) + (z+1) \right\} \cup \left\{ \left(\frac{k-1}{2}\right) - (z+1), \left(\frac{k-1}{2}\right) - (z+2), \right.$$
$$\left. \left(\frac{k-1}{2}\right) - (z+3), \ldots, 3, 2, 1 \right\}$$

From the above it is clear that the vertex labels assigned to the b vertices in A and d vertices in B are distinct. Since $A \cap B = \phi$ and $A \cup B = \{1, 2, \ldots, k-3, k-2\}$. Similarly the induced edge labels in the set A' and B' are distinct. Since $A' \cap B' = \phi$ and $A' \cup B' = \{1, 2, \ldots, k-3, k-2\}$.
From the above labeling of T^*, we have

$$|V_i(T^*)| = |V_j(T^*)|, \text{ for } 1 \leq i < j \leq k-2 \text{ and} \tag{7}$$

$$|E_i(T^*)| = |E_j(T^*)|, \text{ for } 1 \leq i < j \leq k-2. \tag{8}$$

We have, from (1) and (2)

$$||V_0(T)| - |V_{k-1}(T)|| \leq 1 \text{ and}$$
$$||E_0(T)| - |E_{k-1}(T)|| \leq 1$$

From (1), (2), (7) and (8), we have $||V_i(T^*)| - |V_j(T^*)|| \leq 1$ and $||E_i(T^*)| - |E_j(T^*)|| \leq 1$, for all i, j, $0 \leq i < j \leq k-1$. Thus, T^* is k-equitable tree containing the given arbitrary tree T.

Case 1.2. b **is even and** d **is odd**

Then we can write $d = 2w + 1$, where w is a positive integer. Then assign the d labels belonging to the set C to the d vertices of the remaining $(b + d)$ vertices.

$$C = \left\{ \left(\frac{k-1}{2}\right) - w, \left(\frac{k-1}{2}\right) - (w-1), \left(\frac{k-1}{2}\right) - (w-2), \ldots, \right.$$
$$\left. \left(\frac{k-1}{2}\right) + (w-2), \left(\frac{k-1}{2}\right) + (w-1), \left(\frac{k-1}{2}\right) + r \right\}$$

Then the induced edge labels of the pendant edges having an end vertex in S_2 and the other end vertex having the label $k-1$ in T contained in T^* are given in the following set C'.

$$C' = \left\{ \left(\frac{k-1}{2}\right) + w, \left(\frac{k-1}{2}\right) + (w-1), \left(\frac{k-1}{2}\right) + (w-2), \ldots, \right.$$
$$\left. \left(\frac{k-1}{2}\right) - (w-2), \left(\frac{k-1}{2}\right) - (w-1), \left(\frac{k-1}{2}\right) - w \right\}$$

Assign the b labels belonging to the set D to the b vertices of the remaining $(b + d)$ vertices.

$D = \{1, 2, \ldots, k-3, k-2\} \backslash C$. That is,
$$D = \left\{ 1, 2, 3, \ldots, \left(\frac{k-1}{2}\right) - (w+3), \left(\frac{k-1}{2}\right) - (w+2), \right.$$
$$\left. \left(\frac{k-1}{2}\right) - (w+1) \right\} \cup \left\{ \left(\frac{k-1}{2}\right) + (w+1), \left(\frac{k-1}{2}\right) + (w+2), \right.$$
$$\left. \left(\frac{k-1}{2}\right) + (w+3), \ldots, k-4, k-3, k-2 \right\}$$

Then the induced edge labels of the pendant edges having an end vertex in S_2 and the other end vertex having the label 0 in T, contained in T^*, are given in the following set D'.

$$D' = \left\{ 1, 2, 3, \ldots, \left(\frac{k-1}{2}\right) - (w+3), \left(\frac{k-1}{2}\right) - (w+2), \right.$$
$$\left. \left(\frac{k-1}{2}\right) - (w+1) \right\} \cup \left\{ \left(\frac{k-1}{2}\right) + (w+1), \left(\frac{k-1}{2}\right) + (w+2), \right.$$
$$\left. \left(\frac{k-1}{2}\right) + (w+3), \ldots, k-4, k-3, k-2 \right\}$$

We observe from the label sets C and D, the vertex labels assigned to the d vertices having the labels in the set C and the vertex labels assigned to the b vertices having the labels in the set D are distinct, since $C \cap D = \phi$ and $C \cup D = \{1, 2, \ldots, k-2\}$. Similarly the induced edge labels in the set C' and D' are distinct, since $C' \cap D' = \phi$ and $C' \cup D' = \{1, 2, \ldots, k-2\}$.

From the above labeling of T^*, we have

$$|V_i(T^*)| = |V_j(T^*)| \text{ for } 1 \leq i < j \leq k-2 \text{ and} \tag{9}$$

$$|E_i(T^*)| = |E_j(T^*)| \text{ for } 1 \leq i < j \leq k-2. \tag{10}$$

We have, from (1) and (2)

$$||V_0(T)| - |V_{k-1}(T)|| \leq 1 \quad \text{and}$$
$$||E_0(T)| - |E_{k-1}(T)|| \leq 1$$

From (1), (2), (9) and (10), we have $||V_i(T^*)| - |V_j(T^*)|| \leq 1$ and $||E_i(T^*)| - |E_j(T^*)|| \leq 1$, for all i, j, $0 \leq i < j \leq k-1$. Thus, T^* is k-equitable containing the given arbitrary tree T.

Case 2. When k is even,

Then $k-2$ is even. As $b + d = k-2$, $b + d$ is even. This would imply that either both b and d are even or both b and d are odd. By the construction of T^*, when k is even m and n are chosen to be even. Therefore from the equation (3) we have $m - a(k-2) = b$ is also even. Similarly from the equation (4) we have $n - c(k-2) = d$ is also even. Thus both b and d cannot be odd. Consequently we consider only the case both b and d are even. Let $b = 2h$, $h \in Z^+$.

Now assign the b labels belonging to the set L to the b vertices of the remaining $(b+d)$ vertices.

$$L = \left\{\left(\frac{k-2}{2}\right) - (h-1), \left(\frac{k-2}{2}\right) - (h-2), \left(\frac{k-2}{2}\right) - (h-3), \ldots, \left(\frac{k-2}{2}\right) + (h-2), \left(\frac{k-2}{2}\right) + (h-1), \left(\frac{k-2}{2}\right) + h\right\}$$

Then the induced edge labels of the pendant edges having one end vertex in S_2 and the other end vertex having the label 0 in T are given in the following set L'.

$$L' = \left\{\left(\frac{k-2}{2}\right) - (h-1), \left(\frac{k-2}{2}\right) - (h-2), \left(\frac{k-2}{2}\right) - (h-3), \ldots, \left(\frac{k-2}{2}\right) + (h-2), \left(\frac{k-2}{2}\right) + (h-1), \left(\frac{k-2}{2}\right) + h\right\}$$

Assign the d labels belonging to the set M to the d vertices of the remaining $(b+d)$ vertices.

$$\begin{aligned} &M = \{1, 2, \ldots, k-2\}\backslash L. \text{ That is,} \\ &M = \left\{1, 2, 3, \ldots, \left(\frac{k-2}{2}\right) - (h+2), \left(\frac{k-2}{2}\right) - (h+1), \left(\frac{k-2}{2}\right) - h\right\} \\ &\quad \cup \left\{\left(\frac{k-2}{2}\right) + (h+3), \left(\frac{k-2}{2}\right) + (h+2), \right. \\ &\quad \left. \left(\frac{k-2}{2}\right) + (h+1), \ldots, (k-4), (k-3), (k-2)\right\} \end{aligned}$$

Then the induced edge labels of the pendant edges having one end vertex in F (which is a 1-degree vertex) and the other end vertex having the label $k-1$ in T are given in the following set M'.

$$\begin{aligned} M' = &\left\{k-2, k-3, k-4, \ldots, \left(\frac{k}{2}\right) + (h+2), \frac{k}{2} + (h+1), \frac{k}{2} + r\right\} \\ &\cup \left\{\left(\frac{k}{2}\right) - (h+3), \left(\frac{k}{2}\right) - (h+2), \left(\frac{k}{2}\right) - (h+1), \ldots, 3, 2, 1\right\} \end{aligned}$$

We observe from the set L and M, the vertex labels assigned to b vertices in the set L and the vertex labels assigned to d vertices in M are distinct. Since $L \cup M = \{1, 2, \ldots, k-2\}$ and $L \cap M = \phi$. Similarly the induced edge labels in the set L' and M' are distinct, since $L' \cap M' = \phi$ and $L' \cup M' = \{1, 2, \ldots, k-2\}$.

From the above labeling of T^*, we have

$$|V_i(T^*)| = |V_j(T^*)|, \text{ for } 1 \leq i < j \leq k-2 \text{ and} \tag{11}$$

$$|E_i(T^*)| = |E_j(T^*)|, \text{ for } 1 \leq i < j \leq k-2. \tag{12}$$

We have, from (1) and (2)

$$\begin{aligned} ||V_0(T)| - |V_{k-1}(T)|| &\leq 1 \quad \text{and} \\ ||E_0(T)| - |E_{k-1}(T)|| &\leq 1 \end{aligned}$$

From (1), (2), (11) and (12), for T^*, we have

$$||V_i(T^*)| - |V_j(T^*)|| \leq 1 \text{ and } ||E_i(T^*)| - |E_j(T^*)|| \leq 1$$

for all i, j, $0 \leq i < j \leq k-1$. Thus, T^* is k-equitable containing the given arbitrary tree T.

Hence the theorem.

Illustration:

We illustrate below the process of obtaining a super k-equitable tree T^* for a given tree T.

Consider the tree T given in Figure 1. Then its supertree T^* constructed as given in the proof of Theorem 2.1 is given in Figure 2. In Figure 3 the cordial labeling of T contained in T^* is given based on the Horse-Race Labeling Algorithm. In Figure 4 the label 1 is changed to the label 8 $(= k-1)$. In Figure 5 the 9-equitable labeling is defined for the supertree T^* (containing the given tree T) as given in the proof of Theorem 2.1.

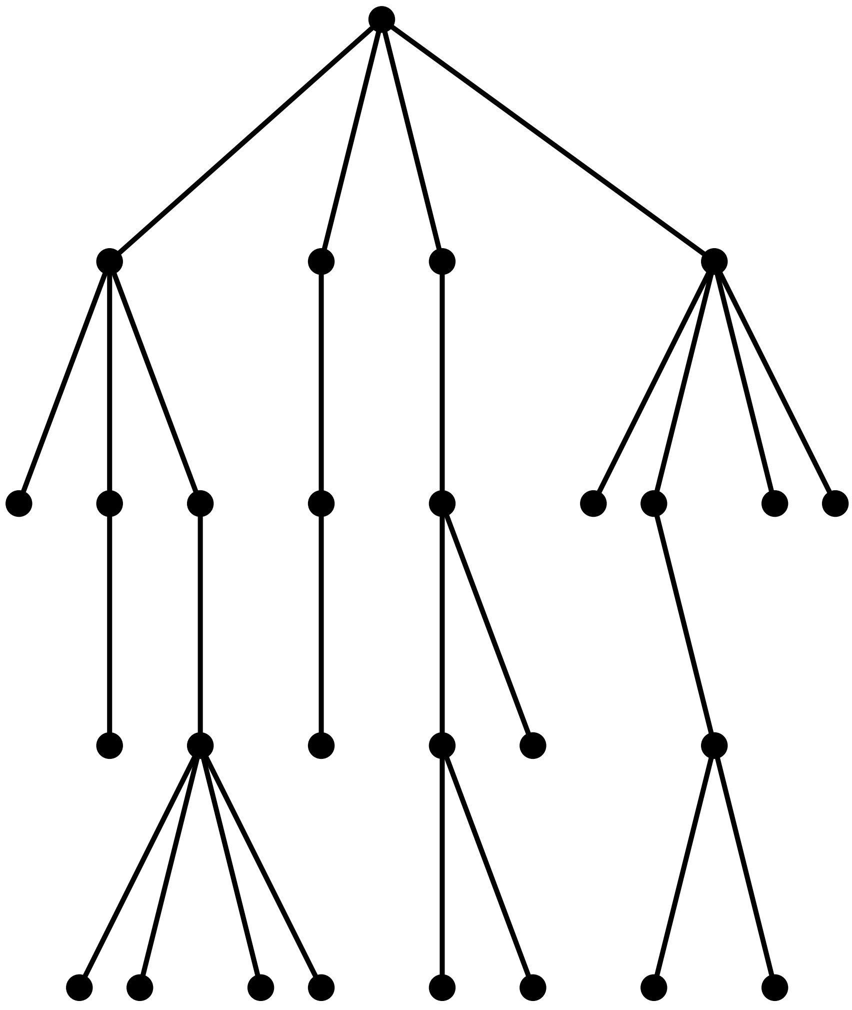

Figure 1: The tree T

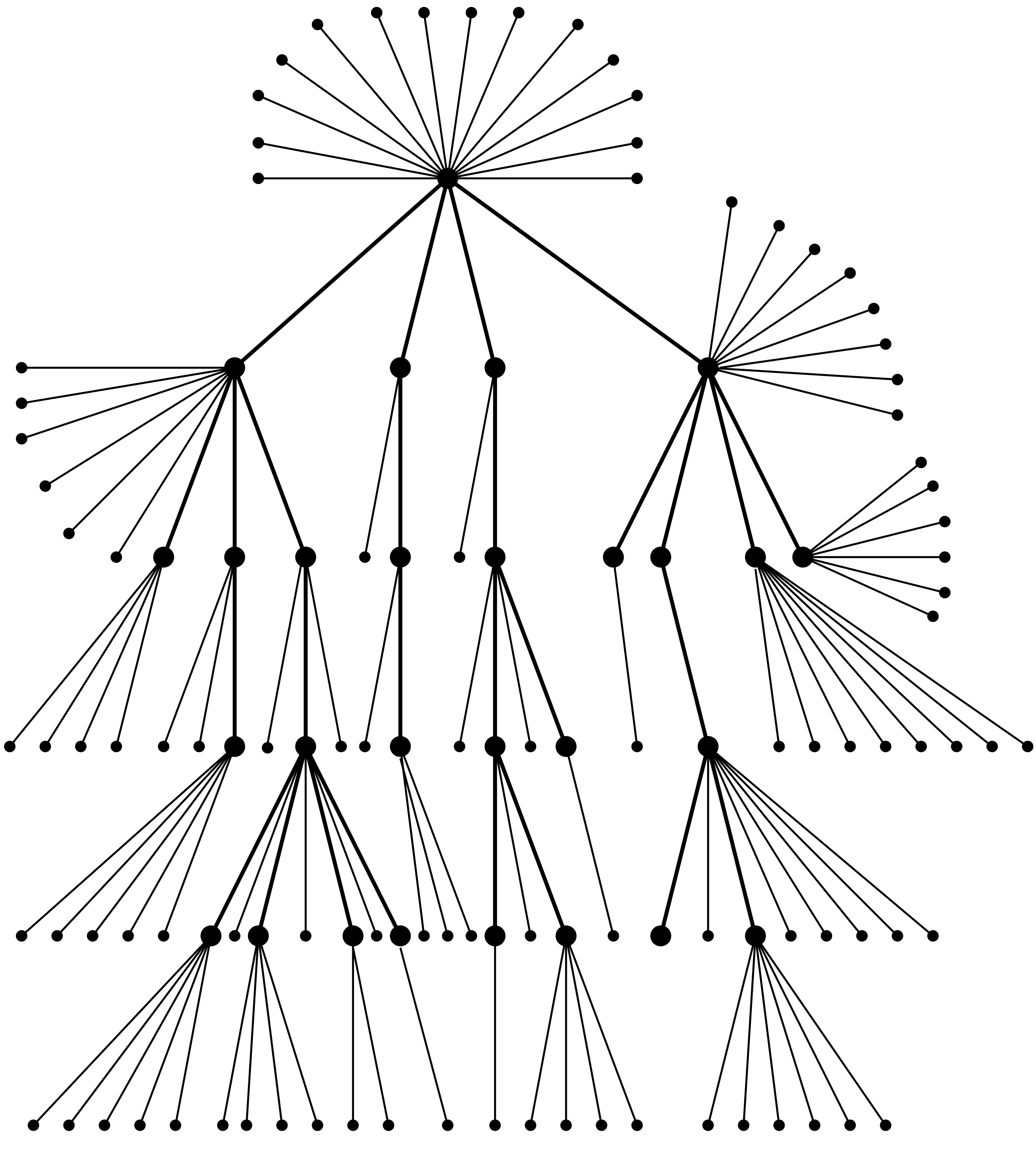

Figure 2: The tree T^*

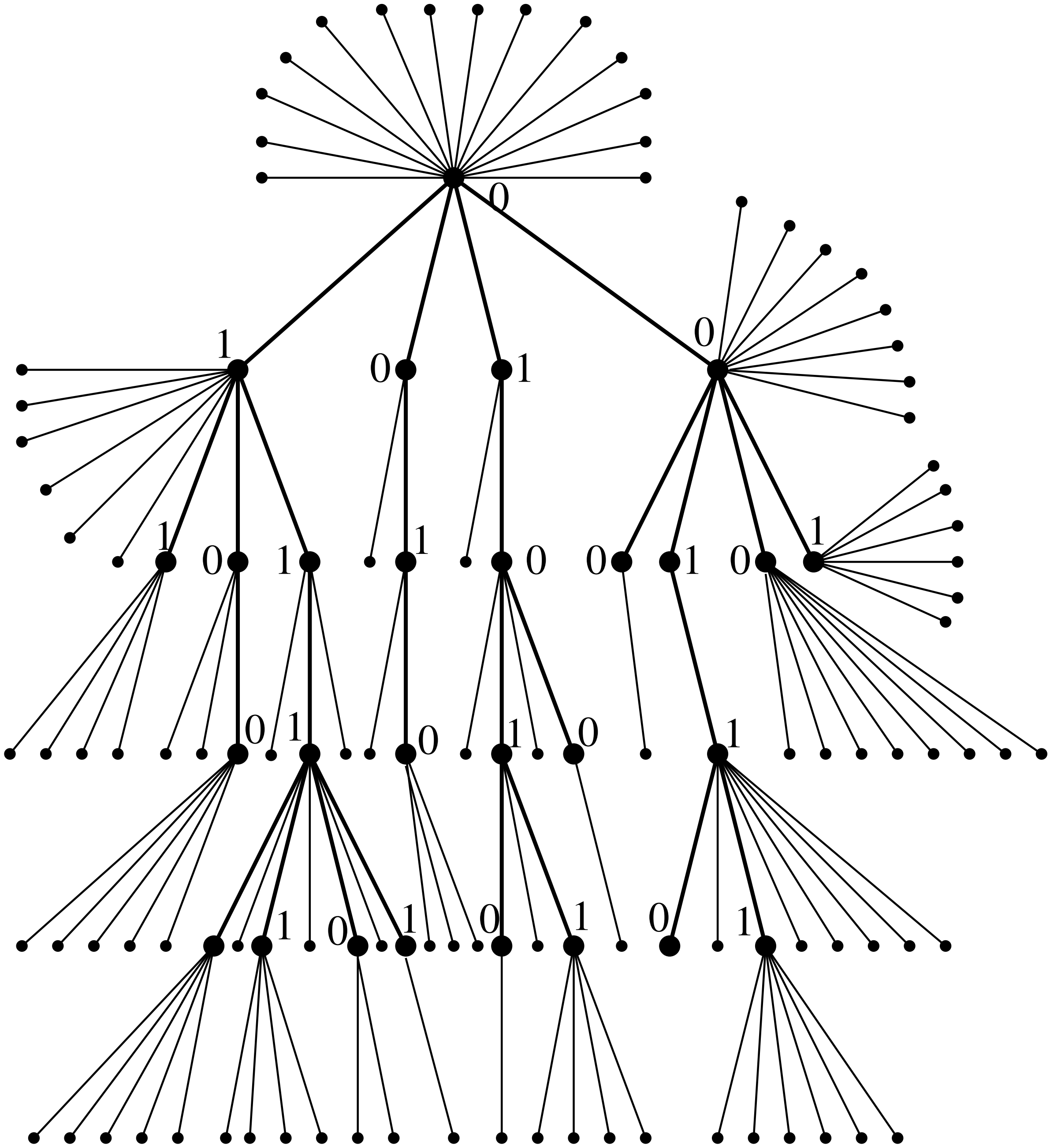

Figure 3: 0-1 labeling of the tree T contained in T^*

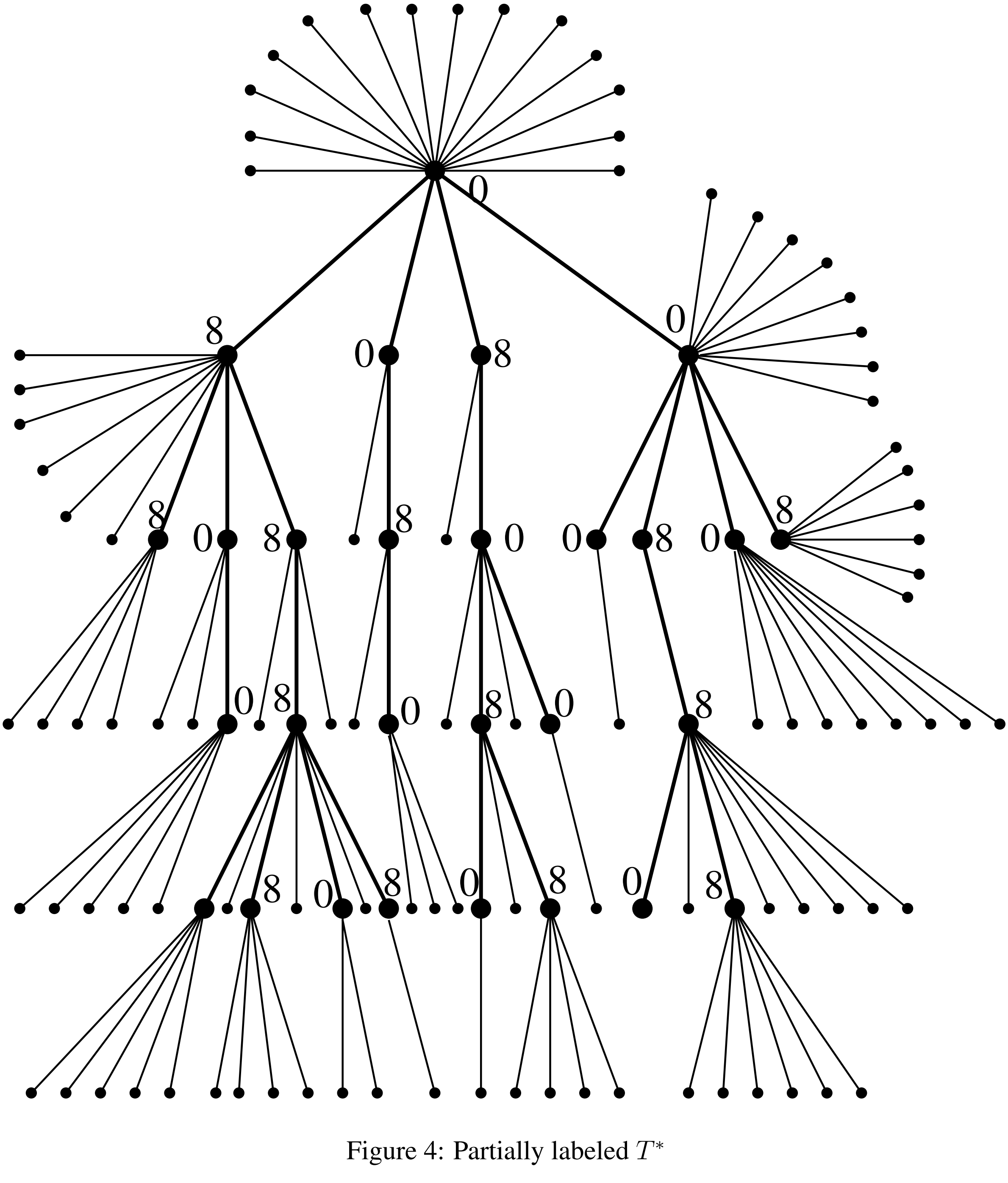

Figure 4: Partially labeled T^{*}

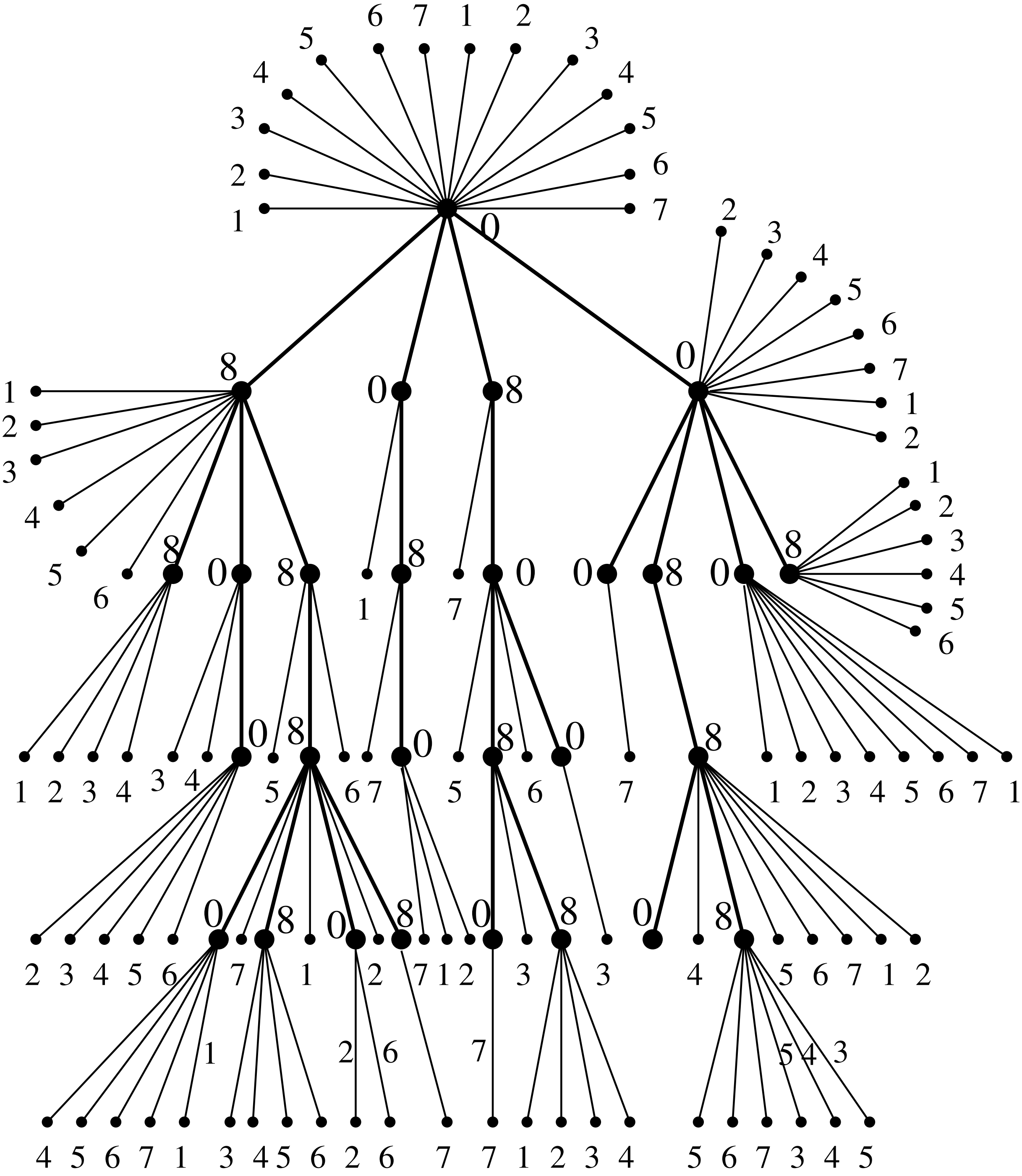

Figure 5: 9-equitable labeled tree T^* containing the given tree T

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