Even Graceful Labeling of the Union of Paths and Cycles

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Abstract

M. Ibrahim Moussa proved that the graph $C_m \cup P_n$ is odd graceful if $m$ is even, he also described an algorithm to label the vertices and the edges of the vertex set $V(C_m \cup P_n)$ and the edge set $E(C_m \cup P_n)$. In this paper we studied the graph $C_m \cup P_n$ is even graceful if $m$ is odd with some certain condition.

Keywords: Graceful labeling, graceful graph, even graceful labeling.

Introduction

If the vertices are assigned values subject to certain conditions then it is known as graceful labeling. The study of graceful graphs and graceful labeling methods was introduced by Rosa [1].

A function $f$ is called graceful labeling of graph $G$ if $f: V \rightarrow \{0, 1, 2, \ldots, q\}$ is injective and the induced function $f^*: E \rightarrow \{1, 2, \ldots, q\}$ defined as $f^*(uv) = |f(u) - f(v)|$ is bijective. A graph which admits graceful labeling is called graceful graph.

A graph $G$ of size $q$ is odd graceful or even graceful if there is an injection $f$ from $V(G)$ to $\{0,1,2,3,4,\ldots,2q-1\}$ such that when each edge $uv$ is assigned the label $|f(u) - f(v)|$, the resulting edge labels are $\{1,3,5,\ldots,2q-1\}$ or $\{2,4,6,\ldots,2q-2\}$ respectively.

Even Graceful of $C_m \cup P_n$

Theorem

If $K$ is a given integer and $m = 2k+1$, then the graph $C_m \cup P_n$ is even graceful for every $n = m+2$. 
Proof
Let \( V(C_m) = \{u_1, u_2, \ldots, u_m\} \), \( V(P_n) = \{ v_1, v_2, \ldots, v_n\} \) where \( V(C_m) \) is the vertex set of the cycle \( C_m \) and \( V(P_n) \) is the vertex set of path \( P_n \) and \( q = (n+m) \).

For every \( u_i \) and \( v_i \) we defined the even graceful labeling functions \( f(u_i) \) and \( f(v_i) \) respectively as follows:

**Vertex labeling of \( C_m \)**

- \( f(u_1) = 0 \)
- \( f(u_2) = 2q - 2 \)
- \( f(u_3) = 2 \)
- \( f(u_4) = 2q - 4 \)
- \( f(u_5) = 4 \)
- \( f(u_6) = 2q - 6 \) ....
- \( f(u_{k+1}) = \text{zero and even} \)
- \( f(u_k) = 2q - i \)

**Vertex labeling of \( P_n \)**

\[
 f(v_i) = \begin{cases} 
 i & \text{if } i = 0, 2, 4, \ldots, \\
 (q+2) - i & \text{if } i = 0, 2, 4, \ldots, 
\end{cases}
\]

The first value of \( f(v_i) \) denoted the labeling of the vertex which are given below and the second value of \( f(v_i) \) denoted the labeling of above vertices of the path \( P_n \).

The number of vertices in the path \( P_n \) arranged as follows: The number of above vertices is \((k+2)\) and the number of below vertices is \((k+1)\).

If \( k \) is even each \((C_m \cup P_n)\) graph we just neglect the labeling \((6+n), n = 2, 4, 6, \ldots\) in the path \( P_n \). If \( k \) is odd we eliminate \((6+n), n= \text{multiples of 6}\) from each \((C_m \cup P_n)\) graph.

**Edge labeling of \( C_m \)**

- \( f^*(u_1u_2) = 2q - 2 \)
- \( f^*(u_2u_3) = 2q - 4 \)
- \( f^*(u_3u_4) = 2q - 6 \)
- \( f^*(u_{m-1}u_m) = 2q - 2p \)
- \( f^*(u_mu_{i+1}) = 2q - 4m + p, \text{ where } p=0, 2, 4, \ldots \)

**Edge labeling of \( P_n \)**

- \( f^*(v_iv_{i+1}) = 2q - 2m - p, \ldots \) except the labeling \((2q - 4m + p)\) since that labeling belongs to the \( C_m \).

These conditions shows that each component in the given graph has even graceful, the path is even graceful, the cycle with an odd number of vertices is even graceful. We have to prove that the vertex labels are distinct and all the edge labels are even numbers \( \{2, 4, 6, \ldots, 2q - 2\} \).

In this theorem, the conditions which are mentioned above for odd and even value of \( k \), to prove the distinction of vertex and edge labeling. Otherwise it cannot form the graceful graph.

This theorem is illustrated by the following examples.

**Example**

Let \( V(C_5) = \{u_1, u_2, u_3, u_4, u_5\} \), \( V(P_5) = V(P_{m+2}) = V(P_7) = \{ v_1, v_2, v_3, v_4, v_5, v_6, v_7\} \)
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Where $V(C_5)$ is the vertex set of the cycle $C_5$ and $V(P_7)$ is the vertex set of the path $P_7$ and $q = n+m = 12$. Number of vertices above the path is 3 and below the path 4. The even graceful labeling functions $f(u_i)$ and $f(v_i)$ as given below.

$$
f(u_1) = 0 \quad f(u_2) = 22 \quad f(u_3) = 2 \quad f(u_4) = 20 \quad f(u_5) = 4
$$

Here we neglect the vertex labeling 6 from the path $P_7$.

$$
f(v_1) = 0 \quad f(v_2) = 14 \quad f(v_3) = 2 \quad f(v_4) = 12 \quad f(v_5) = 4
$$

$$
f(v_6) = 10 \quad f(v_7) = 8
$$

The edge labeling function $f^*$ defined as follows,

$$
f^*(u_1u_2) = 22 \quad f^*(u_2u_3) = 20 \quad f^*(u_3u_4) = 18
$$

$$
f^*(u_4u_5) = 16 \quad f^*(u_5u_1) = 4.
$$

and

$$
f^*(v_1v_2) = 14 \quad f^*(v_2v_3) = 12 \quad f^*(v_3v_4) = 10
$$

$$
f^*(v_4v_5) = 8 \quad f^*(v_5v_6) = 6 \quad f^*(v_6v_7) = 2
$$

Clearly we conclude that all the vertex labeling and edge labeling are even graceful labeling. Therefore this graph is even graceful graph.

**Example**

Consider the graph $(C_4 \cup P_6)$ where $m = 4$ and $q = 10$. Using the above theorem the vertex labeling of the cycle is $f(u_1) = 0 \quad f(u_2) = 18 \quad f(u_3) = 2 \quad f(u_4) = 16$. The corresponding edge labeling is $f^*(u_1u_2) = 18 \quad f^*(u_2u_3) = 16 \quad f^*(u_3u_4) = 14 \quad f^*(u_4u_1) = 16$
Clearly the above cycle is not graceful and the above and below arrangement of the path $P_n$ is not suitable for this path $P_6$.
Therefore the theorem is only true for the odd value of $m$.

**Conclusion**

In this paper, we explicitly proved a theorem of even graceful labeling of the graph $(C_m \cup P_n)$ for $m$ is odd. Also we found this theorem is not true for $m$ is even.

**References**