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Simultaneous Measurement of Non-commuting Variables with appropriate Post-Selection in Weak measurement

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Abstract

Determinism is non-trivial in quantum mechanics as the interaction of the measurement apparatus with the measured system is central to its formulation. The observing apparatus disturbs the observed system in an unpredictable and uncontrolled manner that sets limitation on the precision of the measurement which is the feature of the well-known Heisenberg Uncertainty Principle. In an ideal measurement also known as 'projective' measurement, the relative shifts in the pointer corresponding to different eigenstates of 'X' are large compared to the initial uncertainty in the pointer's position. This results in an unambiguous separation of the overlap of different eigenstates of 'X' known as the 'collapse of the wave function'. This is the idea underlying a strong measurement following the von Neumann interaction model for the measurement of the observable 'X'. The collapse of the wave function precludes the simultaneous measurement of the non-commuting observables such as position and momentum.

In a seminal work Aharanov, Albert and Vaidman (1988) proposed a new technique known as 'weak measurements' which entailed weakening the interaction between the measuring apparatus and the measured system to prevent the collapse mentioned above. The technique involved pre-selection, weak interaction and post-selection to give rise to 'weak value' of an observable. These weak values are complex and possess interesting features. We shall review the technique in this paper to propose schemes to simultaneously measure non-commuting variables. We shall show that, weak values may be used to reconstruct the wave function 'ψ' of the quantum system. This method is the quantum state tomography using weak measurements.

Keywords: Non-commuting observables, weak measurement, post-selection, weak value, tomography.

INTRODUCTION

Measurement is very profound in the formulation of physical laws. One can measure the initial state of the particle viz. the position and the momentum to a very high precision depending on the measuring device. The time evolution in the Hamiltonian or the Lagrangian formulation leads to determinism in classical mechanics. However, things are quite different in quantum mechanics. In the quantum realm of things, the interaction between the measuring apparatus and the measured system disturbs the system in an unpredictable and uncontrolled manner leading to decoherence or collapse of the wave function. This renders the simultaneous measurement of non-commuting observables untenable and non-trivial. Traditionally, this is known as the projective measurement which is consistent with the Heisenberg Uncertainty Principle. The uncertainty is a consequence of this intrinsic feature of quantum measurement and does not depend on the precision of the measuring device. In 1988, Aharanov, Albert and Vaidman conceived the idea of 'weak measurement' in a seminal paper [1]. The proposed idea was reviewed critically by Duck, Stevenson and Sudarshan [2] and applied extensively in different experiments as a laboratory tool [3], [4], [5], [6], [7]. The idea provided a new kind of value for a quantum variable known as the 'weak value' of an observable. In this paper we examine the interesting features of weak measurement in the light of quantum measurement and study its application in quantum tomography.

WEAK MEASUREMENTS

To start with, let us understand the process of quantum measurement. The interaction of the measuring pointer with the system is central to the formulation of measurement theory in quantum mechanics. Following von Neumann's interaction model for the measurement of an observable 'A', the measurement procedure consists of coupling the observable A of the quantum system to a measuring pointer by a coupling Hamiltonian $\tilde{H}$ = g(t)·P·A and let Q, P refer to the position and conjugate momentum of the pointer respectively, g(t) is a function with compact support near the time of the measurement. Hence the strong interaction between the device and the system lasts for so short a time that the changes of the device and the system under observation that would have taken place in the absence of interaction can be neglected. Thus, at least while the interaction is taking place we can neglect the parts of the Hamiltonian associated with the device alone and with the observed system alone and we need to retain only the part of the Hamiltonian representing the interaction. The measurement pointer is initially a gaussian wave function centered at zero.

$$\langle x|\varphi\rangle = \varphi\,(x) = \left(\frac{1}{\sigma\sqrt{2\pi}}\right)^{\frac{1}{2}}\, exp\left(-\,\frac{x^2}{4\sigma^2}\right) \tag{1}$$

where σ is the standard deviation of the probability distribution $|\varphi(x)|^2$.

Let the quantum system be prepared in some quantum state $|\psi_i\rangle$. We consider an ensemble of systems for the purpose of measurement and therefore the state preparation is also termed as pre-selection. Under the action of the coupling Hamiltonian, the whole system of the pointer and the quantum system evolves to

$$|\psi_f\rangle = exp\left(\frac{-iHt}{\hbar}\right) |\psi_i\rangle \; |\varphi_i\rangle = \left(1 - \frac{iHt}{\hbar} - \cdots\right)|\psi_i\rangle \; |\varphi_i\rangle \tag{2}$$

$$= |\psi_i\rangle\,|\varphi_i\rangle - \frac{igt}{\hbar}\hat{A}\,|\psi_i\rangle \;\; \hat{P}|\varphi_i\rangle - \ldots \tag{3}$$

The measurement process consists of shifting the mean position of the pointer by an amount proportional to the expectation value of the observable $\langle A\rangle$ after the interaction. In an ideal measurement, the relative shifts corresponding to the different eigenvalues of A are large compared to the initial uncertainty in the pointer's position. This results in unambiguous separation of the overlap of the different eigenstates of A known as 'decoherence' or the 'collapse of the wave function'. Thus it is not possible to measure simultaneously the non-commuting variables using an ideal measurement technique. This is the wisdom underlying conventional measurement or strong measurements. This for an ideal measurement: (i) The measurement always produces one of its eigenvalues a_n . (ii) The probability of its outcome a_n is $|c_n|^2$ where $|\psi_i\rangle = \sum_n c_n \;|A = a_n\rangle$ expressed in terms of its eigenstates. Thus for a projective measurement, the pointer is left in a state consisting of widely separated spikes each centered on one of its eigenvalues a_n. In the case of the weak measurement, we may approximate it to a single broad gaussian peaked at the mean value of $\hat{A}$ as the interaction between the apparatus and the quantum system is considered weak.

Immediately, after the weak measurement of A, one makes a strong measurement by restricting to the sub ensemble of system states that are found to be in $|\psi_f\rangle$. This procedure of projecting out the part of the state is known as post-selection. Such a 'weak' measurement technique results in expectation values which are remarkably different than in strong or projective measurements. On post-selecting in state $|\psi_f\rangle$, the measurement state will be left in the state as follows,

$$|\varphi_f\rangle = \langle\psi_f|\; exp\left(\frac{-iHt}{\hbar}\right) |\psi_i\rangle \;\; |\varphi_i\rangle$$

$$= \langle\psi_f|\psi_i\rangle\,|\varphi_i\rangle - \frac{igt}{\hbar}\langle\psi_f|\;\hat{A}\,|\psi_i\rangle \;\; \hat{P}|\varphi_i\rangle - \ldots \tag{4}$$

On normalizing the state and considering only the terms up to the first order of g we obtain,

$$|\varphi_f\rangle = |\varphi_i\rangle - \frac{igt}{\hbar}\frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}\;\hat{P}|\varphi_i\rangle - \ldots \tag{5}$$

The expression $\frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}$ is the weak value of $\hat{A}$. (6)

The expectation value of position $\hat{X}$ of the pointer is calculated as

$$\langle\hat{X}\rangle = \langle\varphi_f|\hat{X}|\varphi_f\rangle = \left(\langle\varphi_i| + \frac{igt}{\hbar}\frac{\langle\psi_i|\hat{A}|\psi_f\rangle}{\langle\psi_i|\psi_f\rangle}\langle\varphi_i|\hat{P}\ |\ \hat{X}\ |\ \varphi_i\rangle - \frac{igt}{\hbar}\frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}\hat{P}|\varphi_i\rangle \right)$$

$$= \langle\varphi_i|\hat{X}|\varphi_i\rangle + \frac{igt}{\hbar}\frac{\langle\psi_i|\hat{A}|\psi_f\rangle}{\langle\psi_i|\psi_f\rangle}\langle\varphi_i|\hat{P}\hat{X}|\varphi_i\rangle - \frac{igt}{\hbar}\frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}\langle\varphi_i|\hat{X}\hat{P}|\varphi_i\rangle \quad (7)$$

Now the first term is zero. The other terms are rewritten as,

$$\langle\hat{X}\rangle = \frac{-igt}{\hbar} Re\left(\frac{\langle\psi_f|A|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}\right)\langle\varphi_i|\hat{X}\hat{P} - \hat{P}\hat{X}|\varphi_i\rangle + \frac{gt}{\hbar} Im\left(\frac{\langle\psi_f|A|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle}\right)\langle\varphi_i|\hat{X}\hat{P} + \hat{P}\hat{X}|\varphi_i\rangle$$

We use the integrals $\int_{-\infty}^{+\infty} e^{-bx^2}dx = \sqrt{\frac{\pi}{b}}$ and $\int_{-\infty}^{+\infty} e^{-bx^2}\, x^2 dx = \frac{1}{2b}\sqrt{\frac{\pi}{b}}$ to evaluate $\langle\varphi_i|\hat{X}\hat{P} + \hat{P}\hat{X}|\varphi_i\rangle = 0$. Finally we obtain,

$$\langle\hat{X}\rangle = \langle\varphi_f|\hat{X}|\varphi_f\rangle = gt \times \text{Real part of } \frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} \quad (8)$$

Similarly the expectation value of momentum is calculated as

$$\langle\hat{P}\rangle = \langle\varphi_f|\hat{P}|\varphi_f\rangle = \frac{\hbar gt}{2\sigma^2} \times \text{Im } \frac{\langle\psi_f|\hat{A}|\psi_i\rangle}{\langle\psi_f|\psi_i\rangle} \quad (9)$$

The weak value of the observable may be expressed in terms of the two expectation values of the pointer,

$\langle\hat{A}\rangle_w =$ Re $\langle\hat{A}\rangle_w$ + i Im $\langle\hat{A}\rangle_w$.

The weak value is a complex number and we have seen that the real part gives the position of the pointer and the imaginary part gives us the momentum. It was also shown that the weak values can lie far beyond the range of the observable's eigenvalues [1].This is known as weak value amplification and is used effectively in several applications [3], [4], [5].

DETERMINING THE STATE OF THE QUANTUM SYSTEM

The above discussion facilitates the measurement of the non-commuting variables viz. position and momentum with the weak measurement technique. The real part and the imaginary part of the weak value corresponds to the variable and its complementary counterpart. This enables us to measure the wave function ψ of the quantum system which represents the state of the system. At each x, the observed position and momentum shifts of the measurement pointer are proportional to Re $\langle\hat{A}\rangle_w$ and Im $\langle\hat{A}\rangle_w$. Scanning the weak measurement through x thus entails re-construction of the wave function by the simultaneous measurement of non-commuting variables. This is the quantum state tomography using weak measurements.

CONCLUSIONS

The weak values are complex nature with the real and imaginary part having physical significance [8],[9]. Interesting Quantum effects arise on appropriate Post-Selection. This have been illustrated for the measurement of position and momentum for a quantum system. Further, this facilitates the simultaneous measurement of non-commuting variables which entails reconstruction of the wave function [10]. The weak measurement holds potential for quantum tomography with enhanced photon counting using appropriate post-selection and weak value amplification.

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