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On the Principle of Exchange of Stabilities in Rayleigh-Benard Convection in a Porous Medium with Variable Gravity by Positive Operator Method

Dr. Pushap Lata Sharma

Assistant Professor Deptt of Mathematics Govt. College,
Arki, Himachal Pradesh, India.

Abstract

In the present paper, the problem of Rayleigh –Benard Convection in porous medium heated from below with variable gravity is analyzed and it is established by the method of positive operator of Weinberger and uses the positivity properties of Green's function that principle of exchange of stabilities is valid for this general problem, when g(z) is non-negative throughout the fluid layer.

INTRODUCTION

Rayleigh–Bénard convection is a fundamental phenomenon found in many atmospheric and industrial applications. The problem has been studied extensively experimentally and theoretically because of its frequent occurrence in various fields of science and engineering. This importance leads the authors to explore different methods to study the flow of these fluids. Many analytical and numerical methods have been applied to analyze this problem in the domain of Newtonian fluids, including the linearized perturbation method, the lattice Boltzmann method (LBM), which has emerged as one of the most powerful computational fluid dynamics (CFD) methods in recent years.

A problem in fluid mechanics involving the onset of convection has been of great interest for some time. The theoretical treatments of convective problems usually invoked the so-called principle of exchange of stabilities (PES), which is demonstrated physically as convection occurring initially as a stationary convection. This has been stated as "all non decaying disturbances are non oscillatory in time". Alternatively, it can be stated as "the first unstable eigenvalues of the linearized system has imaginary part equal to zero".

Pellew and Southwell (1940) took the first decisive step in the direction of the establishment of PES in Rayleigh-Benard convection problems in a comprehensive manner. S. H. Davis (1969) proved an important theorem concerning this problem. He proved that the eigenvalues of the linearized stability equations will continue to be real when considered as a suitably small perturbation of a self-adjoint problem, such as was considered by Pellew and Southwell. This was one of the first instances in which Operator Theory was employed in hydrodynamic stability theory. As one of several applications of this theorem, he studied Rayleigh-Benard convection with a constant gravity and established PES for the problem. Since then several authors have studied this problem under the varying assumptions of hydromagnetic and hydrodynamics.

Convection in porous medium has been studied with great interest for more than a century and has found many applications in underground coal gasification, solar energy conversion, oil reservoir simulation, ground water contaminant transport, geothermal energy extraction and in many other areas. In the present paper, the problem of Rayleigh –Benard Convection in porous medium heated from below with variable gravity is analyzed by the method of positive operator and it is established that principle of exchange of stabilities valid for this general problem, when g(z) is non-negative throughout the fluid layer.

2. Mathematical Formulation of the Physical Problem

Consider an infinite horizontal porous layer of fluid of depth‘d’ confined between two horizontal planes $z = 0$ and $z = d$ under the effect of variable gravity, $\vec{g}(0,0-g(z))$. Let ΔT be the temperature difference between the lower and upper plates. Thus, the governing equations for the Rayleigh-Benard flow–saturated porous medium under Boussinesq approximation and under the effect of variable gravity are;

$$\frac{1}{\varepsilon}\left[\frac{\partial \vec{q}}{\partial t}+\frac{1}{\varepsilon}(\vec{q}.\nabla)\vec{q}\right]=-\frac{\nabla p}{\rho_0}+\left(1+\frac{\delta\rho}{\rho_0}\right)\vec{X} \tag{1}$$

$$\nabla.\vec{q}=0 \tag{2}$$

$$E\frac{\partial T}{\partial t}+(\vec{q}.\nabla)T=K\nabla^2 T \tag{3}$$

$$\rho=\rho_0\left[1-\alpha\left(T-T_0\right)\right] \tag{4}$$

In the above equations, $\vec{q}$, T ρ, K, α and υstand for filter velocity, temperature, density, thermal diffusivity, coefficient of thermal expansion, and the kinematic viscosity, respectively. Here, E=$\varepsilon+(1-\varepsilon)\frac{\rho_s c_s}{\rho_0 c_v}$ is a constant, where ρ_s, c_s stand for density and heat capacity of solid (porous matrix) material and ρ_0, c_v for fluid, respectively. Here, the suffix zero refers to the value at the reference level z=0.

Following the usual steps of the linearized stability theory, it is easily seen that the nondimensional linearized perturbation equations governing the physical problem described by equations (1)-(4) can be put into the following forms, upon ascribing the dependence of the perturbations of the form

$exp\left[i\left(k_x x + k_y y\right) + \sigma t\right]$, $(\sigma = \sigma_r + i\sigma_i)$

$$\left[\frac{\sigma}{\varepsilon} + \frac{1}{p_l}\right](D^2 - k^2)w = -g(z)R\,\theta k^2 \tag{5}$$

$$\left(D^2 - k^2 - \sigma E P_r\right)\theta = -Rw \tag{6}$$

together with following dynamically free and thermally and electrically perfectly conducting boundary conditions

$$w = 0 = \theta = D^2 w \text{ at } z = 0 \text{ and } z = 1 \tag{7}$$

In the forgoing equations, z is the real independent variable, $D \equiv d/dz$ is the differentiation with respect to z, k^2 is the square of the wave number, $\Pr = \frac{\upsilon}{\kappa}$ is the thermal Prandtl number, $P_l = \frac{k_1}{d^2}$ is the dimensionless medium permeability. E= $\varepsilon + (1-\varepsilon)\frac{\rho_s c_s}{\rho_0 c_v}$ is constant, $R_T = R^2 = \frac{g_0 \alpha\beta d^4}{\kappa\upsilon}$ is the thermal Rayleigh number, $\sigma(= \sigma_r + i\sigma_i)$ is the complex growth rate associated with the perturabations and w, θ are the perturbations in the vertical velocity, temperature, respectively.

The system of equations (5)-(6) together with the boundary conditions (7) constitutes an eigenvalue problem.

3. ABSTRACT FORMULATION

THE METHOD OF POSITIVE OPERATOR

We seek conditions under which solutions of equations (5)-(6) together with the boundary conditions (7) grow. The idea of the method of the solution is based on the notion of a 'positive operator', a generalization of a positive matrix, that is, one with all its entries positive. Such matrices have the property that they possess a single greatest positive eigenvalue, identical to the spectral radius. The natural generalization of a matrix operator is an integral operator with non-negative kernel. To apply the method, the resolvent of the linearized stability operator is analyzed. This resolvent is in the form of certain integral operators. When the Green's function Kernels for these operators are all nonnegative, the resulting operator is termed positive. The abstract theory is based on the Krein –Rutman theorem (1962), which states that;

"If a linear, compact operator A, leaving invariant a cone $\hbar$, has a point of the spectrum different from zero, then it has a positive eigen value λ, not less in modulus than every other eigen value, and this number corresponds at least one eigen vector

$\phi \in \hbar$ of the operator A, and at least one eigen vector $\varphi \in \hbar^*$ of the operator A^*". For the present problem the cone consists of the set of nonnegative functions.

To apply the method of positive operator, formulate the above equations (5) and (6) together with boundary conditions (7) in terms of certain operators as;

$$\left(\frac{\sigma}{\varepsilon}+\frac{1}{p_l}\right)\tilde{M}w = g(z)R\theta k^2 \tag{8}$$

$$\left(\tilde{M}+\sigma E\,Pr\right)\theta = Rw \tag{9}$$

where, $\tilde{M}w = mw, \quad w \in dom\tilde{M}; \quad \tilde{M}^2 w = m^2 w, \quad w \in dom\left(\tilde{M}\tilde{M}\right);$ and $\tilde{M}\theta = m\theta, \quad w \in dom\tilde{M}$

The domains are contained in B, where

$$B = L^2(0,1) = \left\{\phi \;\middle|\; \int_0^1 |\phi|^2\, dz < \infty\right\},$$

with scalar product $\langle\phi,\varphi\rangle = \int_0^1 \phi(z)\,\overline{\varphi(z)}\,dz$, $\phi,\varphi \in B$; and norm $\|\phi\| = \langle\phi,\phi\rangle^{1/2}$.

We know that $L^2\,(0,1)$ is a Hilbert space, so, the domain of M is

dom $\tilde{M}$ = $\{\phi \in B / \; D\phi, m\phi \in B, \;\; \phi(0) = \phi(1) = 0\}$.

We can formulate the homogeneous problem corresponding to equations (5)-(6) by eliminating θ from (8) and (9) as;

$$w = k^2 R_T \tilde{M}^{-1}\left(\frac{\sigma}{\varepsilon}+\frac{1}{p_1}\right)^{-1} g(z)\left(\tilde{M}+E\,Pr\,\sigma\right)^{-1} \tag{10}$$

or

$$w = K(\sigma)w, \tag{11}$$

where,

$$K(\sigma) = K^2 R_T \tilde{M}^{-1}\left(\frac{\sigma}{\varepsilon}+\frac{1}{p_1}\right)^{-1} g(z)\left(\tilde{M}+E\,Pr\,\sigma\right)^{-1} \tag{12}$$

But $T(E\,Pr\,\sigma) = \left(\tilde{M}+E\,Pr\,\sigma\right)^{-1}$ and exists for $\sigma \in T_{\frac{k}{\sqrt{Pr\,E}}} = \left\{\sigma \in C \;\middle|\; Re(\sigma) > \frac{-k^2}{E\,Pr}, Im(\sigma) = 0\right\}$ and $\|T(\sigma E\,Pr)\|^{-1} > \left|\sigma + \frac{k^2}{E\,Pr}\right|$ for Re $(\sigma) > -\frac{k^2}{E\,Pr}$.

Now, $T(E\,Pr\,\sigma)$ is an integral operator such that for $f \in B$,

$$T(\sigma E\,\mathrm{Pr})f = \int_0^1 g(z,\xi;\sigma E\,\mathrm{Pr})f(\xi)d\xi$$
,

where, $g(z,\xi,E\,\mathrm{Pr}\,\sigma)$ is Green's function kernel for the operator $(M+\sigma E\,\mathrm{Pr})$, and is given as

$$g(z,\xi,\mathrm{Pr}\,\sigma) = \frac{\cosh[r(1-|z-\xi|)] - \cosh[r(-1+z+\xi)]}{2r\sinh r}$$

where, $r = \sqrt{k^2 + \sigma E\,\mathrm{Pr}}$.

In particular, taking $\sigma = 0$, we have $M^{-1} = T(0)$ is also an integral operator.

$K(\sigma)$ defined in (12), which is a composition of certain integral operators, is termed as linearized stability operator. K(σ) depends analytically on σ in a certain right half of the complex plane. It is clear from the composition of K(σ) that it contains an implicit function of σ .

We shall examine the resolvent of the K(σ) defined as $[I-K(\sigma)]^{-1}$

$$[I-K(\sigma)]^{-1} = \{I-[I-K(\sigma_0)]^{-1}[K(\sigma)-K(\sigma_0)]\}^{-1}[I-K(\sigma_0)]^{-1} \tag{13}$$

If for all σ_0 greater than some a,

(1) $[I-K(\sigma_0)]^{-1}$ is positive,

(2) $K(\sigma)$ has a power series about σ_0 in $(\sigma_0-\sigma)$ with positive coefficients; i.e., $\left(-\frac{d}{d\sigma}\right)^n K(\sigma_o)$ is positive for all n, then the right side of (13) has an expansion in $(\sigma_0-\sigma)$ with positive coefficients. Hence, we may apply the methods of Weinberger (1969) and Rabinowitz (1969), to show that there exists a real eigenvalue σ_1 such that the spectrum of $K(\sigma)$ lies in the set $\{\sigma : \mathrm{Re}(\sigma) \le \sigma_1\}$. This is result is equivalent to PES, which was stated earlier as "the first unstable eigenvalue of the linearized system has imaginary part equal to zero."

4. THE PRINCIPLE OF EXCHANGE OF STABILITIES (PES)

It is clear that $K(\sigma)$ is a product of certain operators. Condition (1) can be easily verified by following the analysis of Herron (2000, 2001) for the present operator $K(\sigma)$. The operator $\tilde{M}^{-1} = T(0)$ is an integral operator whose Green's function $g(z,\xi;0)$ is nonnegative so $\tilde{M}^{-1}$ = T(0) is a positive operator. It is mentioned above that $T(E\,\mathrm{Pr}\,\sigma)$ is an integral operator its Green's function kernel g$(z,\xi,E\,\mathrm{Pr}\,\sigma)$ is the Laplace transform of the Green's function $\frac{1}{E\,\mathrm{Pr}}G\left(z,\xi;\frac{t}{E\,\mathrm{Pr}}\right)$ for the boundary value problem

$$\left(-\frac{\partial^2}{\partial z^2}+k^2+E\Pr\frac{\partial}{\partial t}\right)G=\delta(z-\xi,t) \tag{14}$$

where, $\delta(z-\xi,t)$ is Dirac –delta function in two-dimension, with boundary conditions G$\left(0,\xi;\frac{t}{E\Pr}\right)$ =0= G$\left(1,\xi;\frac{t}{E\Pr}\right)$ and intial condition $G(z,\xi;0)=0$ (15)

Following Herron (2000), by direct calculation of the inverse Laplace transform, we can have

$$\left(-\frac{d}{d\sigma}\right)^n g(z,\xi,\sigma E\Pr)=\int_0^\infty t^n e^{-\sigma E\Pr t}\,G(z,\xi,t)\,dt \geq 0$$ for all n and $\sigma>-\frac{k^2}{E\Pr}$.

$T(\sigma E\Pr)=\left(\tilde{M}+\sigma E\Pr\right)^{-1}$ is positive operator for all real $\sigma_0>-\frac{k^2}{E\Pr}$, and that $T(\sigma E\Pr)$ has a power series for all real $\sigma_0>-\frac{k^2}{E\Pr}$

It has been demonstrated that all of the terms in $K(\sigma)$ determine positive operator. i.e. $K(\sigma)$ is a linear, compact integral operator. Thus, $K(\sigma)$ is a positive Moreover, for σ real and sufficiently large, the norms of the operators $T(0)$, $T(\Pr\sigma)$ become arbitrarily small. So, $\|K(\sigma)\|<1$. Hence, $[I-K(\sigma)]^{-1}$ has a convergent Neumann series, which implies that $[I-K(\sigma)]^{-1}$ is a positive operator. This is the content of condition (1).

To verify condition(2) for g(z)≥ 0 for all $z\in[0,1]$ and $\sigma_0>\max\{-\frac{k^2}{E\Pr}\}$, while k^2 and R_T are clearly positive. Therefore by the product rule for differentiation, one concludes that $K(\sigma)$ in (12) satisfies condition (2).

Conclusions

In the present paper, principle of exchange of stabilities valid for this general problem, when g(z) is non-negative throughout the fluid layer.

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