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General Solutions to Boundary Layer & Related PDEs of Fluid Dynamics

Steve Anglin, Sc.M., Ph.D.(h.c.)
Department of Mathematics,
Case Western Reserve University, Cleveland, OH, USA.

Abstract

The nonlinear boundary layer hydrodynamic partial differential equations (PDEs) typically model laminar, turbulent and other flows of fluid across various geometric surfaces, edges and more used by mechanical engineers, physicists as well as applied and industrial mathematicians. This article covers some practical problem-solution examples of these nonlinear PDEs and some variants through definitions, then corresponding examples. This includes some variations of Navier-Stokes equations and more.

Keywords: Boundary layer, PDEs, partial differential, differential equations, hydrodynamic, Navier-Stokes, fluid, dynamics, mechanics, physics, applied math, nonlinear, non-linear, dispersion, Boussinesq, Ostrovsky, ocean waves, diffusion, heat, mass, transfer, anisotropic, Oseen, vortices, vortex, solution, Coriolis force, Navier-Stokes, Boussinesq, Ostrovsky and other equations. The function *u* is continuous and differentiable (C_2) and serves as the stream or flow function solution in most cases."

Definition

$$u_y u_{xy} - u_x u_{yy} = \nu u_{yyy}$$

$$u(x,y) = \phi(x) exp[-C_1 y] + \nu C_1 x + C_2$$

$$u(x,y) = C_1 y + \phi(x)$$

$$u(x,y) = C_1 y^2 + \phi(x) y + \frac{1}{4C_1}\phi(x)^2 + C_2$$

Definition

$$u_z u_{xz} - u_x u_{zz} = \nu(z u_{zz})_z + f(x)$$

$$u(x,z) = \pm z[2\int f(x)dx + C_1]^{1/2} + \phi(x)$$

$$u(x,z) = \pm z[2\int f(x)dx + C_1]^{1/2} + \phi(x)$$

Example

$$u_z u_{xz} - u_z u_{zz} = \nu(z u_{zz})_z + x^3$$

$$u(x,z) = \pm z[2\int x^3 dx + C_1]^{1/2} + \phi(x)$$

$$u(x,z) = \pm z[\frac{x^4}{2} + C_2 + C_1]^{1/2} + \phi(x)$$

Definition

$$u_y u_{xy} - u_x u_{yy} = k(u_{yy})^{n-1} u_{yyy}$$

$$u = \frac{1}{C_1^2 n(2n-1)}[C_1(n-1)y + C_2]^{(2n-1)/(n-1)}$$

$$+C_3 y + C_4 - kC_1 x$$

Example

$$u_y u_{xy} - u_x u_{yy} = 5(u_{yy})^2 u_{yyy}$$

$$u = \frac{1}{15C_1^2}[2C_1 y + C_2]^{\frac{5}{2}} + C_3 y + C_4 - 5C_1 x$$

Definition

$$u_y u_{xy} - u_x u_{yy} = k(u_{yy})^{n-1} u_{yyy} + f(x)$$

$$u(x,y) = \pm y[2\int f(x)dx + C_1]^{1/2} + \phi(x)$$

Example

$$u_y u_{xy} - u_x u_{yy} = 5(u_{yy})^2 u_{yyy} + 6ex^5$$

$$u(x,y) = \pm y[12e \int x^5 dx + C_1]^{1/2} + \phi(x)$$

Definition

$$u_{zt} + u_z u_{xz} - u_x u_{zz} = \nu(zu_{zz})_z + f(x,t)$$

$$u = C_1 z^2 + \phi z + \frac{\phi^2}{4C_1} + \frac{1}{2C_1}\int \phi_t dx -$$

$$-\frac{1}{2C_1}\int f(x)dx - \nu x + \Psi$$

Example

$$u_{zt} + u_z u_{xz} - u_x u_{zz} = \nu(zu_{zz})_z + 7t^5 x^3$$

$$u = C_1 z^2 + \phi z + \frac{\phi^2}{4C_1} + \frac{1}{2C_1}\int \phi_t dx -$$

$$-\frac{7t^5 x^4}{8C_1} + C_2 - \nu x + \Psi$$

Example

$$u_{zt} + u_z u_{xz} - u_x u_{zz} = \nu(zu_{zz})_z + 2\pi e^x ln(t)$$

$$u = C_1 z^2 + \phi z + \frac{\phi^2}{4C_1} + \frac{1}{2C_1}\int \phi_t dx -$$

$$-\frac{\pi ln(t) e^x}{C_1} + C_2 - \nu x + \Psi$$

Definition

$$u_y u_{xy} - u_x u_{yy} = f(u_{yy})_y + g(x)$$

$$u(x,y) = \pm[2\int g(x)dx + C_1]^{1/2} + \phi(x)$$

Example

$$u_y u_{xy} - u_x u_{yy} = f(u_{yy})_y + 3x^2$$

$$u(x,y) = \pm y[2\int 3x^2 dx + C_1]^{1/2} + \phi(x)$$

$$u(x,y) = \pm y[2x^3 + C_2 + C_1]^{1/2} + \phi(x)$$

Definition: Nonlinear Dispersion in Patterns from Drops PDE

$$u_t = a(u^2)_{xxx} + b(u^2)_x$$

$$u = \frac{2C_2}{3b} \pm \frac{2C_2}{3b} sin[\frac{1}{2}\frac{\sqrt{b}}{\sqrt{a}}(x - C_2 t) + C_1]$$

Example

$$u_t = 4(u^2)_{xxx} + 2(u^2)_x$$

$$u = \frac{C_2}{3} \pm \frac{C_2}{3} sin[\frac{\sqrt{2}}{4}(x - C_2 t) + C_1]$$

Definition

$$u_{xt} + (u_x)^2 - uu_{xx} = f(t)u_{xxx}$$

$$u = \phi(t)exp[-\lambda x] - \frac{\phi'(t)}{\lambda\phi(t)} + \lambda f(t)$$

Example

$$u_{xt} + (u_x)^2 - uu_{xx} = 3t^{3/2}u_{xxx}$$

$$u(x,t) = \phi(t)e^{-\lambda x} - \frac{\phi'(t)}{\lambda\phi(t)} + 3\lambda t^{3/2}$$

Definition

$$u_{xt} + u_{xx} - uu_{xx} = \nu u_{xxx} + f(t)$$

$$f(t) = Ae^t$$

$$u(x,t) = \psi e^{\lambda x} - \frac{Ae^{\beta t - \lambda x}}{4\lambda^2\psi} + \frac{\psi'}{\lambda\psi} - \nu\lambda$$

Example

$$u_{xt} + u_{xx} - uu_{xx} = \nu u_{xxx} + 8e^{3t}$$
$$u = \psi e^{\lambda x} - \frac{2e^{3t-\lambda x}}{\lambda^2 \psi} + \frac{\psi'}{\lambda \psi} - \nu\lambda$$

Example

$$u_{xt} + u_{xx} - uu_{xx} = \nu u_{xxx} + 4\lambda^2 e^{2t}$$
$$u = \psi e^{\lambda x} - \frac{e^{2t-\lambda x}}{\psi} + \frac{\psi'}{\lambda \psi} - \nu\lambda$$

Definition

$$(u_t + u_{xxx} - 6uu_x)_x + au_{yy} = 0$$
$$u(x,y,t) = -2[ln(x + C_1 y - aC_1^2 t)]_{xx}$$

Example

$$(u_t + u_{xxx} - 6uu_x)_x + 4u_{yy} = 0$$
$$u(x,y,t) = -2[ln(x + C_1 y - 4C_1^2 t)]_{xx}$$

Definition

$$u_y u_{xy} - u_x u_{yy} = f(x) u_{yyy}$$
$$u = \theta e^{\lambda y} - \lambda \int f(x)dx + C$$

Example

$$u_y u_{xy} - u_x u_{yy} = 3e^x u_{yyy}$$
$$u = \theta e^{\lambda y} - 3\lambda e^x + C_2 + C_1$$

Example

$$u_y u_{xy} - u_x u_{yy} = 2ln(x) u_{yyy}$$
$$u = \theta e^{\lambda y} - \frac{2\lambda}{x} + C_2 + C_1$$

Definition

$$u_y u_{xxx} - u_x u_{xxy} = f(y) u_x$$

$$u = C_1 e^{C_4 x} + C_2 e^{-C_4 x} + C_3 - \frac{1}{C_4^2} \int f(y) dy$$

$$u = C_1 cos(C_4 x) + C_2 sin(C_4 x) + C_3 - \frac{1}{C_4^2} \int f(y) dy$$

Example

$$u_y u_{xxx} - u_x u_{xxy} = sin(y) u_x$$

$$u = C_1 cos(C_4 x) + C_2 sin(C_4 x) + C_3 - \frac{1}{C_4^2} \int sin(y) dy$$

$$u = C_1 cos(C_4 x) + C_2 sin(C_4 x) + C_3 + \frac{cos(y)}{C_4^2} + C_5$$

Definition: Ostrovsky's Ocean Waves PDE

$$u^2 u_t - u_x u_{xt} + u u_{xxt} = 0$$

$$u(x,t) = 2C_3 - 6C_3 tanh^2[\sqrt{C_3}(x - C_2 t) + C_1]$$

Definition: Boussinesq-Type PDE

$$u_{tt} = u_{xx} + 2a(u u_x)_x + b u_{ttxx}$$

$$u(x,t) = -\frac{1}{2a} + \frac{3b}{2at^2} + \frac{x^2}{2at^2}$$

Example

$$u_{tt} = u_{xx} + 8(u u_x)_x + 8 u_{ttxx}$$

$$a = 4, b = 8$$

$$u(x,t) = -\frac{1}{8} + \frac{3}{t^2} + \frac{x^2}{8t^2}$$

Definition: Un-Normalized Boussinesq PDE

$$u_{tt} = a(uu_x)_x + bu_{xxxx}$$

$$u = \frac{(x + C_1)^2}{a(t + C_2)^2} - \frac{12b}{a(x + C_1)^2}$$

$$u = \frac{3C_5^2}{a} cosh^{-2}[\frac{C_5}{2\sqrt{b}}(x \pm C_5 t) + C_1]$$

Example

$$u_{tt} = 3(uu_x)_x + 2u_{xxxx}$$

$$a = 3, b = 2$$

$$u = \frac{(x + C_1)^2}{3(t + C_2)^2} - \frac{8}{(x + C_1)^2}$$

$$u = C_5^2 cosh^{-2}[\frac{C_5}{2\sqrt{2}}(x \pm C_5 t) + C_1]$$

Definition: Heat / Mass Transfer in Anisotropic Media PDE

$$(ax^n u_x)_x + (by^m u_y)_y = 0$$

$$u(x, y) = C_1[\frac{x^{2-n}}{a(2-n)} - \frac{y^{2-m}}{b(2-m)}] + C_2$$

Example

$$a = 1, b = 3, n = 1, m = 1/2$$

$$(xu_x)_x + (3y^{1/2} u_y)_y = 0$$

$$u(x, y) = C_1[x - \frac{2y^{3/2}}{9}] + C_2$$

Example

$$a = 2, b = 3, n = 4, m = 5$$

$$(2x^4 u_x)_x + (3y^5 u_y)_y = 0$$

$$u(x, y) = C_1[-\frac{1}{4x^2} + \frac{1}{9y^3}] + C_2$$

Navier-Stokes PDEs

Example: simple diffusion form

$$u_t = \nu(u_{yy} + u_{zz}) + \frac{\Delta p}{L}$$

$$u(y,z,t) = \frac{\Delta p}{4\nu L}(4\pi^2 - (y^2 + z^2))$$

Example: with Oseen Vortices solution

$$\omega_t + v * \omega_x = \nu\Delta\omega$$

$$\omega^o(x,t) = \frac{C}{1+t} exp(-\frac{|x|^2}{4\nu(1+t)})$$

Example: A Navier-Stokes System

$$\boldsymbol{u}_t - \Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}$$

$$\nabla\cdot\boldsymbol{u} = 0$$

$$\nu = 1, \Omega = (0,1)^2, \boldsymbol{f} = (0, Ra(1 - y + 3y^2))^T$$

$$\boldsymbol{u} = 0, p = Ra(y^3 - \frac{y^2}{2} + y - \frac{7}{12})$$

Example: Coriolis force

$$\boldsymbol{u}_t - \Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}$$

$$\nabla\cdot\boldsymbol{u} = 0$$

$$\Omega = (0,10)\times(0,1), \nu = 1$$

$$\boldsymbol{u} = \begin{pmatrix}1\\0\end{pmatrix}, p = \omega(\frac{1}{6} - \frac{y^2}{2}) \Rightarrow \boldsymbol{f} = 0$$

Example: A Navier-Stokes System

$$\boldsymbol{u}_t - \Delta\boldsymbol{u} + (\boldsymbol{u}\cdot\nabla)\boldsymbol{u} + \nabla p = \boldsymbol{f}$$

$$\nabla\cdot\boldsymbol{u} = 0$$

$$Re > 0, \nu = 1, \Omega = (-1,1)^2$$

$$\boldsymbol{u} = \begin{pmatrix}-y\\x\end{pmatrix}, p = Re(\frac{x^2+y^2}{2} - \frac{1}{3})$$

Definition: A Navier Stokes PDE

$$u_y(\Delta u)_x - u_x(\Delta u)_y = \nu\Delta\Delta u + f(y)$$

$$u(x,y) = -\frac{1}{2\nu}\int_0^y (y-z)^2\Phi(z)dz + C_1 e^{-\lambda y}$$

$$+C_2 y^2 + C_3 y + C_4 + \nu\lambda x$$

$$\Phi(z) = e^{-\lambda z}\int e^{\lambda z} f(z)dz$$

Example

$$f(y) = e^{\lambda y}$$

$$\Phi(z) = e^{-\lambda z}\int e^{\lambda z}e^{\lambda z}dz = e^{-\lambda z}\int e^{2\lambda z}dz$$

$$\Phi(z) = [\frac{e^{\lambda z}}{2\lambda} + e^{-\lambda z}C_6]$$

$$u = -\frac{1}{2\nu}\int_0^y (y-z)^2[\frac{e^{\lambda z}}{2\lambda} + e^{-\lambda z}C_6]dz$$

$$u = -\frac{1}{2\nu}\int_0^y (y-z)^2[\frac{e^{\lambda z}}{2\lambda} + e^{-\lambda z}C_6]dz$$

$+C_2 y^2 + C_3 y + C_4 + \nu\lambda x$, where

$$\int_0^y (y-z)^2\left(\frac{e^{\lambda z}}{2\lambda} + e^{-\lambda z}A\right)dz =$$

$$-\frac{-4A\lambda + 4A\lambda e^{\lambda(-y)} + \lambda y(\lambda(A(4-2\lambda y)+y)+2) - 2e^{\lambda y} + 2}{2\lambda^4}$$

And, where

$$A = C_6$$

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