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Taylor's Viscous Problem for Centrifugal Stability / Instability in a Couette Flow: Thermal Convection with Rotation Problem

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Abstract

This problem deals with centrifugal stability in a Couette flow between two concentric rotating cylinders. The convection here is created by centrifugal forces overcoming the viscous forces or effects. We will first use the fundamental equations of motion; then decompose the motion into background states with perturbations. This will derive a new set of linearized or perturbed equations of motion. From these new equations, we get normal mode solutions which will give us solutions that must satisfy the boundary conditions for no-slip on cylinder walls and the velocities u, v, w must be zero on the walls for the general case. Next, we focus on the classical problem of interest, the narrow gap approximation, where the re-formulated equations of motion for u and v can be solved using the u = Du = 0, v =0, for x = 0 and 1. Taking the onset of instability/marginal case where sigma = 0, we solve for v. Then, substitute back for both v and u, and solve for some minimum Taylor number T as a function of an eigenvalue a. This will yield the critical Taylor number and eigenvalue for the onset of instability in this problem.

Keywords: Taylor, viscous, stability, instability, fluid mechanics, fluid dynamics, vorticity, equations, PDEs, differential.

This problem deals with centrifugal instability in a Couette flow between concentric rotating cylinders. The convections here are created by centrifugal forces overcoming the viscous forces.

We define the following:

$$u = u(r,\theta,z), v = v(r,\theta,z), w = w(r,\theta,z)$$

$$\frac{D}{Dt} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial r} + w\frac{\partial}{\partial z}$$

$$\Delta = \frac{\partial}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The above gives us the following equations of motion:

$$\frac{Du}{Dt} - \frac{v^2}{r} = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu(\Delta u - \frac{u}{r^2})$$

$$\frac{Dv}{Dt} + \frac{uv}{r} = \nu(\Delta v - \frac{v}{r^2})$$

$$\frac{Dw}{Dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu(\Delta w)$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

Decompose the motion into a background state with perturbations such that we have the following:

$$u = U + u, v = V + v, w = W + w, p = P + p$$

The background state:

$$U = 0, W = 0$$

$$V = V(r) = Ar + \frac{B}{r} \leftarrow \frac{1}{\rho}\frac{\partial P}{\partial r} = \frac{V^2}{r}$$

Where the constants A, B are defined as the following:

$$A = \frac{\Omega_2 R_2^2 - \Omega_1 R_1^2}{R_2^2 - R_1^2}, B = \frac{(\Omega_1 - \Omega_2) R_1^2 R_2^2}{R_2^2 - R_1^2}$$

Where the following means:

- Omega 1 is angular speed of inner cylinder
- Omega 2 is angular speed of outer cylinder
- R1 is radius of inner cylinder
- R2 is the radius of outer cylinder.

Substituting the perturbed background states into the equations of motion above, we get the following:

$$\frac{D(U+u)}{Dt} - \frac{(V+v)^2}{r} = -\frac{1}{\rho}\frac{\partial(P+p)}{\partial r} + \nu(\Delta(U+u) - (\frac{U+u}{r^2}))$$

$$\frac{D(V+v)}{Dt} + \frac{(U+u)(V+v)}{r} = \nu(\Delta(V+v) - (\frac{V+v}{r^2}))$$

$$\frac{D(W+w)}{Dt} = -\frac{1}{\rho}\frac{\partial(P+p)}{\partial z} + \nu(\Delta(W+w)$$

$$\frac{\partial(U+u)}{\partial r} + \frac{(U+u)}{r} + \frac{\partial(W+w)}{\partial z} = 0$$

Inserting background states above

$$U = 0, V = V(r), W = 0$$

Gives us the following:

$$\frac{\partial u}{\partial t} - \frac{2V}{r}v = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu(\Delta u - \frac{u}{r^2})$$

$$\frac{\partial v}{\partial t} + (\frac{\partial V}{\partial r} + \frac{V}{r})u = \nu(\Delta v - \frac{v^2}{r})$$

$$\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu\Delta w$$

$$\frac{\partial u}{\partial r} + \frac{u}{r} + \frac{\partial w}{\partial z} = 0$$

Become the perturbation equations of motion. Rearranging these with

$$D = \frac{d}{dr}, D_* = \frac{d}{dr} + r$$

Gives us these equations:

$$\nu(DD_* - k^2 - \frac{p}{\nu})u + 2\frac{V}{r}v = \frac{\partial \omega}{\partial r}$$
$$\nu(DD_* - k^2 - \frac{p}{\nu})v - (D_*V)u = 0$$
$$\nu(DD_* - k^2 - \frac{p}{\nu})w = -k\omega$$

where

$$\Delta = (\frac{\partial}{\partial r} + \frac{1}{r})\frac{d}{dr} - k^2 = D_*D - k^2 = DD_* + \frac{1}{r^2} - k^2$$
$$D_*w = -kw = D_*u; \omega = \frac{\delta p}{\rho}$$

And, we have the following normal mode solutions:

$$u = e^{pt} u(r) cos(kz), v = e^{pt} v(r) cos(kz), w = e^{pt} w(r) sin(kz)$$

$$\omega = \frac{p}{\rho} * e^{pt} cos(kz)$$

Eliminating w, we get the following:

$$\frac{\nu}{k^2}(D_* D - k^2 - \frac{p}{\nu}) D_* u = \omega$$

Inserting this into above and rearranging, we have the following:

$$\frac{\nu}{k^2}(DD_* - k^2 - \frac{p}{\nu})(DD_* - k^2)u = 2\frac{V}{r}v$$

$$\nu(DD_* - k^2 - \frac{p}{\nu})v = (D_* V)u$$

Also, let these have the following values:

$$k^2 = a^2/R_2^2, \sigma = pR_2^2/\nu$$

Here are new equations of motion:

$$(DD_* - a^2 - \sigma)(DD_* - a^2)u = a^2 \frac{2B}{\nu}(\frac{1}{r^2} + \frac{AR_2^2}{B})v$$

$$(DD_* - a^2 - \sigma)v = \frac{2A}{\nu} R_2^2 u$$

Now, let $(\frac{2AR_2^2}{\nu})u \to u$, then these equations become the following:

$$(DD_* - a^2 - \sigma)(DD_* - a^2)u = -Ta^2(\frac{1}{r^2} - \kappa)v$$

$$(DD_* - a^2 - \sigma)v = u$$

Where

$$T = -\frac{4ABR_2^2}{\nu^2} = \frac{4\Omega_1^2 R_1^4}{\nu^2}\frac{(1-\mu)(1-\mu/\eta^2)}{(1-\eta^2)^2}$$

$$\kappa = \frac{-AR_2^2}{B} = \frac{1-\mu/\eta^2}{1-\mu}$$

And, the solutions must satisfy boundary conditions for no-slip on cylinder walls at 4 = 1 and r = eta. Velocities u, v, w must be zero (0) on the walls for general case:

$$u = v = 0, w = 0; Du = 0, r = 1, r = \eta$$

And, for the narrow gap case, the above equations can be rewritten in the framework of narrow gap approximations

$$(D^2 - a^2 - \sigma)(D^2 - a^2)u = \frac{2\Omega_1 d^2}{\nu}a^2[1-(1-\mu)x]v$$

$$(D^2 - a^2 - \sigma)v = \frac{2Ad^2}{\nu}u$$

Where

$$x = (r - R_1)/d, \kappa = a/d, \sigma = pd^2/\nu$$

Taking this mapping of u as a transformation

$$u \mapsto \frac{2\Omega_1 d^2 a^2}{\nu} u$$

Implies the following equations instead:

$$(D^2 - a^2 - \sigma)(D^2 - a^2)u = (1 + \alpha x)v$$

$$(D^2 - a^2 - \sigma)v = -Ta^2 u$$

Where

$$T = -\frac{4A\Omega_1 d^4}{\nu^2}, \alpha = -(1 - \mu)$$

The above equations can be solved using the following boundary conditions:

$$u = Du = 0; v = 0, x = 1, x = 0$$

Taking Sigma = 0 implies the onset of instability as a stationary secondary flow such that the above equations now become the following:

$$(D^2 - a^2)^2 u = (1 + \alpha x)v$$

$$(D^2 - a^2)v = -Ta^2 u$$

$$u = Du = v = 0, x = 0, x = 1$$

Since v → 0 at x = 0, x = 1, we have the following for v:

$$v = \sum_{m=1}^{\infty} C_m sin(m\pi x)$$

This implies

$$(D^2 - a^2)^2 u = (1 + \alpha x) \sum_{m=1}^{\infty} C_m sin(m\pi x)$$

Then, this implies

$$u = \sum_{m=1}^{\infty} \frac{C_m}{(m^2\pi^2 + a^2)^2} (A_1 cosh(ax) + B_1 sinh(ax) + A_2 x cosh(ax)$$

$$+B_2 x sinh(ax) + (1 + \alpha x) sin(m\pi x) + \frac{4\alpha m\pi}{m^2\pi^2 + \alpha} cos(m\pi x)$$

Where A, B are the constants of integration determined by the boundary conditions. Then, substitute v and u from above to imply the following:

$$\sum_{1}^{\infty} C_n(n^2\pi^2 + a^2) sin(n\pi x) = Ta^2 \sum_{1}^{\infty} \frac{C_m}{(m^2\pi^2 + a^2)^2} (A_1 cosh(ax)$$

$$+B_1 sinh(ax) + A_2 x cosh(ax) + B_2 x sinh(ax) + (1 + \alpha x) sin(m\pi x)$$

$$+\frac{4\alpha m\pi}{m^2\pi^2 + a^2} cos(m\pi x)).$$

This gives us the following form:

$$\frac{1}{2}(\pi^2 + a^2)\frac{1}{Ta^2} = \frac{1}{4}\alpha + \frac{1}{2} - \frac{2a\pi^2(2+\alpha)}{(\pi^2 + a^2) sinh^2 a - a^2}$$

$$*[sinh(a) cosh(a)(ha - a) + (sinh(a) - a cosh(a))].$$

Then, we get the following:

$$T = \frac{2}{2+\alpha}[\frac{(\pi^2+a^2)^3}{1-16a\pi^2 cosh^2(\frac{1}{2}a)/[\pi^2+a^2]^2(sinh(a)+a)}]$$

$$T = (2/(2+\alpha)) * 1,715 = 3,430/(1+\mu)$$

$$a_{min} \simeq 3.12$$

Remark.

This is a minimum of T as a function of a.

Definition.

Taylor's viscous solution indicates the flow remains stable until this critical number is obtained below:

$$T_{cr} = 1,708/[\frac{1}{2}(1+\frac{\Omega_1}{\Omega_2})]$$

$$a_{min} \simeq 3.12$$

The a above is the critical eigenvalue or non-dimensional axial wavenumber at the onset of instability in Couette flow between two rotating, concentric cylinders.

Remark.

T throughout problem is the Taylor number.

References.

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