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Homogeneous Turbulence and the Rate of Viscous Dissipation of Kinetic Energy & Isotropic Turbulence

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Abstract

For homogeneous turbulence, the rate of viscous dissipation of kinetic energy per unit mass is equal to some epsilon function where isotropic turbulence gives us a particular relationship. With dimensional scaling, epsilon has another relationship with some constant c where in principle there exist Reynolds numbers with special relationships.

Keywords*:* homogeneous turbulence, turbulent, isotropic, viscous, function, tensor, energy, fluid dynamics, flow, instability, unstable, velocity, epsilon, Reynolds number, KE, kinetic

For homogeneous turbulence, we show that the rate of viscous dissipation of kinetic energy per unit mass is equal to

$$\epsilon \equiv \frac{1}{2}\nu < (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})^{1/2} >$$

$$= \nu < (\partial u_i / \partial u_j)^2 >$$

Hence, we show that for isotropic turbulence

$$\epsilon = 15\nu(\bar{u_1^2}/\lambda_g^2)$$

Proof:

$$\frac{\partial U_i}{\partial t} + U_j \frac{\partial U_i}{\partial x_j} = -\frac{1}{\rho}\frac{\partial P}{\partial x_i} + \nu \frac{\partial^2 U_i}{\partial x_j \partial x_i}$$

(1)

$$\frac{\partial U_i}{\partial x_j} = 0$$

(Incompressible flow)

Multiply (1) above by u of I and averaging, we get the following:

$$\Rightarrow \rho[\frac{\partial u_i'}{\partial t} + \frac{\partial}{\partial x_j}(\bar{U}_j u_i' + u_j' \bar{U}_i + u_i' u_j' - \bar{u}_i \bar{u}_j)] = \frac{\partial}{\partial x_j}\sigma_{ij}'$$

Where

$$\sigma_{ij}' = -p'\delta_{ij} + 2\mu s_{ij}'$$

And

$$s_{ij}' = \frac{1}{2}(\frac{\partial u_i'}{\partial x_j} + \frac{\partial u_j'}{\partial x_i})$$

$$\Rightarrow (\frac{\partial}{\partial t} + \bar{U}_j \frac{\partial}{\partial x_j})(\frac{1}{2}\rho \bar{u_i'^2}) - \tau_{ij}\frac{\partial \bar{U}_i}{\partial x_j} - \frac{\partial \rho}{\partial x_j}(\frac{1}{2}\bar{u_i'^2}\bar{u_j'})$$

(2)

$$-\frac{\partial}{\partial x_j}(\bar{u_j'}\bar{P'}) + \frac{\partial^2}{\partial x_j^2}(\frac{1}{2}\mu \bar{u_i'^2}) + \frac{\partial}{\partial x_j}(\mu \frac{\partial}{\partial x_i}\bar{u_i'}\bar{u_j'}) - 2\mu s_{ij}'^2$$

where

$$2\mu s'^2_{ij} = -2\mu(\frac{1}{2}(\frac{\partial \bar{u_i}}{\partial x_j} + \frac{\partial \bar{u_j}}{\partial x_i}))(\frac{1}{2}(\frac{\partial \bar{u_i}}{\partial x_j} + \frac{\partial \bar{u_j}}{\partial x_i}))$$

$$= -\mu\frac{1}{2}(\frac{\partial \bar{u_i}}{\partial x_j} + \frac{\partial \bar{u_j}}{\partial x_i})^2$$

$$= \frac{1}{2}\nu < (\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i})^2 > \qquad (3)$$

Now, using

$$\frac{\partial u_i \bar{\partial} u_j}{\partial x_j \partial x_i} = \frac{\partial^2 u_i \bar{u}_j}{\partial x_j \partial x_i}, \frac{\partial u_i}{\partial x_i} = 0$$

$$\Rightarrow \nu\frac{\partial u_i \bar{\partial} u_i}{\partial x_j \partial x_j} + \nu\frac{\partial^2 u_i \bar{u}_j}{\partial x_j \partial x_i} = \nu\frac{\partial u_i \bar{\partial} u_i}{\partial x_j \partial x_j} = \nu(\frac{\partial \bar{u_i}}{\partial x_j})^2$$

$$= \nu < (\frac{\partial u_i}{\partial x_j})^2 >= \epsilon$$

Now, for isotropic turbulence, we get the following:

$$\epsilon = \nu < (\frac{\partial u_i}{\partial x_j})^2 >= 2\nu s'_{ij}\bar{s}_{ij}$$

$$= 2\nu(\frac{\partial u_i}{\partial x_j} \bar{+} \frac{\partial u_j}{\partial x_i})$$

$$= \nu(\frac{\partial u_i}{\partial x_j} \bar{+} \frac{\partial u_j}{\partial x_i})\frac{\partial u_j}{x_i}$$

By isotropy:

$$\Rightarrow (\frac{\partial \bar{u_1}}{\partial x_1})^2 = (\frac{\partial \bar{u_2}}{\partial x_2})^2 = (\frac{\partial \bar{u_3}}{\partial x_3})^2$$

$$\Rightarrow (\frac{\partial \bar{u_1}}{\partial x_2})^2 = (\frac{\partial \bar{u_2}}{\partial x_3})^2 = (\frac{\partial \bar{u_3}}{\partial x_1})^2 = (\frac{\partial \bar{u_1}}{\partial x_3})^2 = (\frac{\partial \bar{u_2}}{\partial x_1})^2 = ...(etc.)$$

$$\Rightarrow \frac{\partial \bar{u_1}}{\partial x_2}\frac{\partial \bar{u_2}}{\partial x_1} = \frac{\partial \bar{u_2}}{\partial x_3}\frac{\partial \bar{u_3}}{\partial x_2} = \frac{\partial \bar{u_1}}{\partial x_3}\frac{\partial \bar{u_3}}{\partial x_1} = ...(etc.)$$

$$\Rightarrow \epsilon = 6\nu[(\frac{\partial \bar{u_1}}{\partial x_1})^2 + (\frac{\partial \bar{u_1}}{\partial x_2})^2 + (\frac{\partial \bar{u_1}}{\partial x_2}\frac{\partial \bar{u_2}}{\partial x_1})]$$

Where the following go to zero in above, by [1.] Hinze, p. 189.

$$(\frac{\partial \bar{u_1}}{\partial x_2})^2 \rightarrow 0, (\frac{\partial \bar{u_1}}{\partial x_2}\frac{\partial \bar{u_2}}{\partial x_1}) \rightarrow 0$$

$$\Rightarrow \epsilon = -15\nu u'^2 \Rightarrow 15\nu(\frac{\partial \bar{u_1}}{\partial x_1})^2 = 15\nu < (\frac{\partial u_1}{\partial x_1})^2 >$$

And, where

$$< (\frac{\partial u_1}{\partial x_1})^2 >= \bar{u_1^2}/\lambda^2 = U^2/\lambda^2$$

$$\Rightarrow \epsilon = 15\nu(U^2/\lambda^2)$$

$$AU^3/L = 15\nu(U^2/\lambda^2)$$

$$\Rightarrow \epsilon = 15\nu(\bar{u_1^2}/\lambda^2) = 15\nu({u'_1}^2/\lambda_g^2)$$

Remark.

$$\Rightarrow \epsilon \simeq 30\nu({u'_1}^2/\lambda_f^2)$$

Part 2, now we have the following:

$$Re_\lambda = u'_1\lambda_g/\nu, Re_L = u'_1 L_f/\nu, Re_\lambda = (Re_L)^\alpha$$

From the simple energy budget:

$$-\bar{u_i}\bar{u_j}S_{ij} = 2\nu\bar{s_{ij}}\bar{s_{ij}}$$

$$\Rightarrow CULS_{ij}S_{ij} = 2\nu\bar{s_{ij}}\bar{s_{ij}}$$

If S of I and j is of order U/L, then the –u of I u of j bar above is of order U squared

$$\Rightarrow \frac{CU^3}{L} = c\frac{\nu U^2}{\lambda^2}$$

The ratio lambda over L or the following Reynolds Number:

$$\Rightarrow Re_\lambda^{-1} \sim \frac{\lambda}{L} = (\frac{c}{C})^{1/2}(\frac{UL}{\nu})^{-1/2} = (\frac{c}{C})^{1/2}Re_L^{1/2}$$

Where alpha = -1/2. Therefore,

$$\therefore Re_\lambda = (Re_L)^{-1/2}$$

For Reynolds Number of Lambda, Lambda over L of f varies

$$\Rightarrow \frac{\lambda_g}{L_f} = (\frac{c}{C})^{1/2}Re_L^{-1/2} = \frac{c}{C}Re_\lambda^{-1}$$

For Reynolds Number of Lambda, Eta of k over Lambda of g varies with c/C = A, constant):

$$\Rightarrow \frac{\lambda_g}{\eta_k} = A^{1/4} \, Re_L^{1/4} = A^{1/4} \, Re_\lambda^{1/2}$$

$$\Rightarrow \frac{\eta_k}{\lambda_g} = A^{-1/4} \, Re_\lambda^{-1/2}.$$

Compliance with Ethical Standards

The author declares that they have no conflict of interest.

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